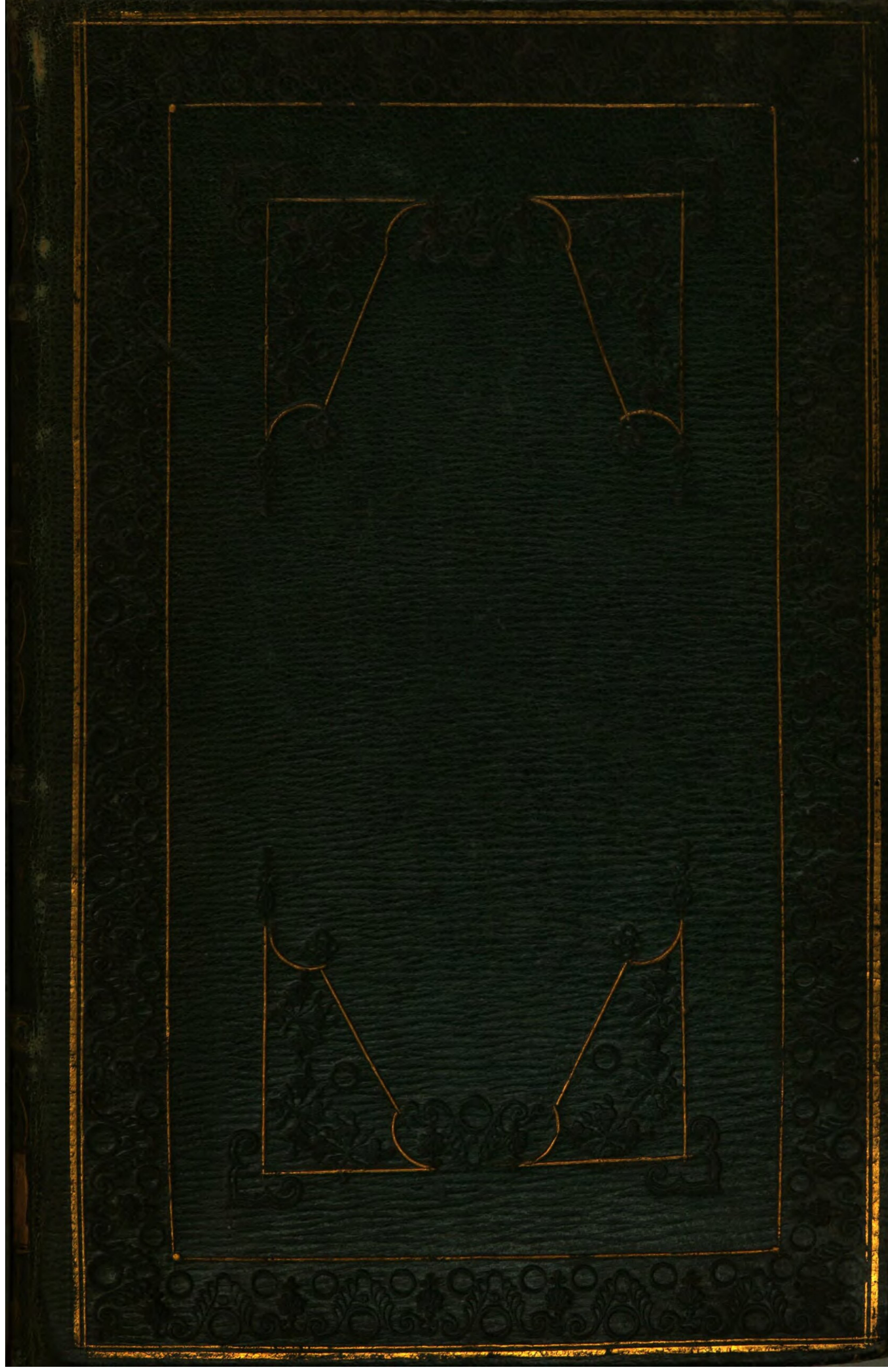






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GEOMETRY.

ELEMENTS

TRIGONOMETRY.

OF

GEOMETRY.

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DUBLIN.

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THE ELEMENTS

OF

GEOMETRY

WITH

ILLUSTRATIONS

AND

TRIGONOMETRY

BY

ADAM SMITH

OF THE UNIVERSITY OF GLASGOW

GEOMETRY

1813

PRINTED

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GLASGOW AND ROBERTSON TO THE F. CO. MAYNORTH

1813

ELEMENTS
OF
GEOMETRY,
WITH BOTH
PLANE AND SPHERICAL
TRIGONOMETRY.

DESIGNED FOR THE USE OF
The Students of the R. C. College, Maynooth.

BY THE
REV. ANDRÉ DARRE.
PROFESSOR OF NATURAL AND EXPERIMENTAL PHILOSOPHY IN THAT
COLLEGE.

DUBLIN:
PRINTED BY H. FITZPATRICK, 4, CAPEL-STREET,
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TO THE
STUDENTS

OF THE
Royal
CATHOLIC COLLEGE OF ST. PATRICK,
AT MAYNOOTH.

GENTLEMEN,

FROM a diffidence in my own abilities, and the consciousness of an imperfect acquaintance with the English phraseology, I always rejected the idea of publishing my Classical Treatises. Your repeated and urgent entreaties, Gentlemen, have at length overcome my reluctance. I have commenced by the present small volume, containing the Elements of Geometry, and of both Plane and Spherical Trigonometry; with the intention of enlarging it, at some future period, and making it a complete Elementary book on Mathematics, by the addition of the Elements of the various branches of Arithmetical and Algebraical Calculation, and the Conic Sections; all indispensibly necessary in the very beginning of my yearly Lectures. My Treatises on the
different

different branches of Physics, which I have composed for the use of, and adapted to the established course of Studies in the College, shall also be published in succession.

Your solicitations, Gentlemen, having induced me to engage in this undertaking; and your voluntary subscriptions for the present volume having supplied the only means of defraying the expense of its publication, I consider it as belonging to you no less than to myself. I, therefore, request you will accept the Dedication of it, as a proof of the sincerity of my wishes to second your application and laudable exertions in the attainment of science, of the zeal I always felt for the prosperity of the College, and of the respectful attachment with which I have the honour to be

GENTLEMEN,

Your most humble,

and obedient servant,

ANDRÉ DARRE.

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 The Reader is requested, before perusing the Work, to make the Corrections pointed out in the Errata placed at the end.

ELEMENTS
OF
GEOMETRY.

1. **GEOMETRY** is the science of *extension*, and of its measurement.

Extension consists of three dimensions, length, breadth, and depth, or thickness. Each of the three considered separately is but a *line*: two, length and breadth, make a *surface*: the three together make a *solid*.

A *point* is an infinitely small part of extension, having no sensible measurable magnitude; and therefore cannot be the object of Geometry: but it is the ultimate element of extension.

A *line* may be considered as formed by the motion of a point, or as a series of contiguous points. So, points are the elements of a line, which therefore has no sensible breadth or depth.

A *surface* may be considered as formed by the motion of a line, or as a series of contiguous lines, as many as there are points in its breadth. So, lines are the immediate; points, the ultimate elements of a surface, which therefore has no sensible thickness.

A *solid* may be considered as formed by the motion of a surface, or as a series of surfaces superimposed
B contiguously

contiguously to one another, as many as there are points in its depth. So, surfaces are the immediate ; lines, the secondary ; points, the ultimate elements of solidity, which therefore has all the three dimensions.

Wherefore Geometry is to be divided into three sections : 1. Of Lines. 2. Of Surfaces. 3. Of Solids.

SECTION I.

OF LINES.

A LINE is *straight* or *curve*. There are curves of various kinds ; none, except the circle, is the object of elementary Geometry. As the measurement of a straight line is merely mechanical, Geometry considers lines comparatively. Compared together, lines are straight or curve, parallel or concurring ; and these meet perpendicularly or obliquely, and make *angles* when there are only two ; *Figures*, when three or more enclose a space. Hence this section is to be subdivided into five heads, viz. 1. The straight and curve lines. 2. The circle. 3. The perpendicular, oblique and parallel lines. 4. The angles. 5. The figures.

Straight and Curve Lines.

2. DEFINITION. A straight line is that of which the elementary points run in the same direction. A line is curve, the elementary points of which change continually their direction.

Such is the most accurate notion Geometry can give of a straight and curve line ; and it is adequate to its object, though not perhaps a logical definition.

Sciences

Sciences mostly begin by such simple ideas sufficiently clear, independent of a definition; and they are no less necessary than self-evident principles of an obvious perspicuity, independent of a demonstration. For want of such simple notions and self-evident principles, an interminable series of definitions and demonstrations should be required; our mind could find no ground whereon to rest in *analysis*, or wherefrom to step in *synthesis*; nothing could be accurately understood, nothing rigorously demonstrated; and a full conviction could never be obtained in the pursuits of sciences.

3. COROLLARIES. 1. Any two points determine invariably the position of a straight line: and three points at least are requisite for the position of a curve or angular line.

2. From one given point to another there can be but one straight line; and two straight lines cannot have more than one common point of meeting or intersection.

3. The straight line is the shortest between two given points; and curve or angular lines are longer as they remove from the straight line.

These three corollaries are self-evident principles. It is obvious that AB (Fig. 1.) is the only possible straight line from the point A to the point B: that there is a greater number of equal elementary points in ADB, or AFB, than in AB; in AEB or AGB than in ADB or AFB: that curve or angular lines joining A and B may increase to infinity on both sides of the straight line AB, as they remove from it; so that the straight line AB is the limits of the decreases of all the curve or angular lines, and, of course, the shortest from A to B.

4. The

4. The straight line is the true measure of distance between two given points. Because the straight line alone is of an invariable length and position.

In practical geometry a rule answers the straight line. A surface or figure is said to be a *plane*, when it is in contact with every point of a straight line in all directions.

The Circle.

4. DEFINITIONS. A *circle* is a plane space contained within a curve line called its *circumference*, every point of which is equidistant from the same inside point, called its *centre*. In practical geometry a circle is described by means of compasses.

A straight line from the centre to any point of the circumference is a *radius*. All the radii of the same circle are equal, as they measure equal distances.

A straight line passing through the centre, and terminated on both sides at the circumference, is a *diameter*. It is equal to two radii. Its extremities are said to be diametrically opposite to each other.

Any part of the circumference is an *arc*; and a straight line joining the two extremities of an arc is a *chord*. A diameter may be considered as a chord, and is the longest of all, since (Fig. 2.) $MCD = AB > MD$.

A straight line, such as TS, out of the circle, but touching the circle only at one point B, or coinciding only with an infinitely small arc, which may be considered as a point, is a *tangent*; B is the *point of contact*.

A chord produced on one side out of the circumference is a *secant*.

The space contained within two radii and their intermediate arc is a *sector*; and the space between a chord and the arc it sustains is a *segment*.

Unequal

Unequal circles which have a common centre are called *concentric circles*; their radii have a constant difference.

5. As an infinitely small arc may be taken for a straight line, with which it coincides sensibly, the circumference of a circle, or any other curve, may be considered as consisting of an infinite number of infinitely small straight lines, inclined to one another; and their tangents are the prolongations of such lines. In the circular curve such lines are equal and equally inclined, otherwise the radii could not be equal; and this is what constitutes the uniform curvature of the circle, its peculiar difference with other curves.

6. On account of that uniformity of curvature, the circumference of the circle may be divided into any number of equal parts; which is of the utmost utility in all geometrical sciences, for the measurement of angles. The division generally adopted from the earlier age of geometry is into 360 equal arcs, called *degrees*; the degree is subdivided into 60 equal parts, called *minutes*; the minute, into 60 *seconds*; the second, into 60 *thirds*; and so on for further subdivisions, when necessary. Mr. Laplace, in his celebrated *Mechanique Celeste*, lately published, has introduced a new division of the circle into 400 degrees, the degree into 100 minutes, the minute into 100 seconds, and so on; which affords a readier change of their parts into decimal fractions. We shall adhere to the ancient division of 360 degrees, and the above subdivisions, designed as follows: $57^{\circ} 45' 36'' 15''' 30''''$ means 57 degrees, 45 minutes, 36 seconds, 15 thirds, 30 fourths, &c.

Radii drawn to each of the 360 divisions of the circumference will divide the circle and all the concentric circles into as many sectors, of one degree each. All the sectors under one degree, or under the
same

same part of a degree, will be equal on every concentric circle; Because the two radii and arc of each being equal, the sectors would coincide, if superimposed to one another; and the successive radii containing equal arcs, will diverge equally from each other. The divergency of two radii, or of any two straight lines, meeting at one of their extremities, is termed in geometry an *angle*.

7. THEOREM 1. The arc of a circle contained by two radii, is a proper measure of their divergency or angle.

DEMONSTRATION. Any two contiguous sectors under equal arcs are equal. Therefore the sum of two is a double sector, under a double arc, with a double divergency of the radii; the sum of three is a triple sector, under a triple arc, with a triple divergency or angle of its radii; and so on for any number of contiguous sectors, under equal arcs. Whence, the arcs contained by two radii are proportional to their divergency or angle, and, of course, a proper measure of it.

The measure of an angle may be taken on any of the concentric circles, since the arcs contained by two radii are of the same number of degrees on every concentric circle (6). Though the degrees of unequal circles are not of the same absolute length, they are of the same angular magnitude or curvature.

8. THEOREM 2. A diameter bisects the circle and its circumference.

DEM. If the figure be folded on the diameter AB, its two arcs ADB, AMB must coincide and be equal; otherwise the arc ADB should fall inside or outside the arc AMB; and as C remains their common centre, the two parts of the same circle should have unequal radii, contrary to the essence of the circle.

COROLL.

9. COROLL. 1. An angle may encrease from 0° to 180° the limits of divergency, which vanishes at its maximum, when two radii forming a diameter become the same straight line.

2. The sum of the two angles made by a radius with the two opposite sides of a diameter, such as $ACN + BCN$ is constant of 180° . And the same may be said of the two angles made by a line with the two sides of another line.

3. When a diameter, such as NG intersects another diameter AB , the two vertical or opposite angles are equal, that is $ACN = BCG$ and $ACG = BCN$. For, as NG is a diameter, it bisects the circle as well as AB (9) and $ACN + BCN = BCN + BCG = 180^\circ$; wherefore $ACN = BCG$. Likewise $BCN + BCG = ACG + BCG = 180^\circ$; and $BCN = ACG$.

4. The same may be said of the vertical angles made by any two straight lines, which intersect each other. Because if a circle be drawn from the point of their intersection as a centre, the two lines become two diameters, and their vertical angles must be equal.

5. A chord divides the circle into two unequal segments, the centre being in the greater.

6. Chords of equal arcs are equal, and reciprocally; because equal arcs, and, of course, their chords should coincide, if superimposed.

7. However chords are not proportional to their arcs. For if the chord and arc $AL = AN$, the chord LN of the double arc is not double of AL , since it is less than $AL + AN$ (3). Nevertheless, greater arcs have evidently greater chords, and reciprocally, $MD > LN$.

Parallel

Parallel Lines.

10. DEFINITION. Two lines are parallel, if, being on the same plane, they have no divergency.

COROLL. Parallel lines have all their corresponding points equidistant (we call corresponding points those which should coincide if the two lines were superimposed, such as A with C, B with D, (Fig. 3.) in the supposition of $AB = CD$). The reason is obvious. The distance from A to C cannot be greater or less than from B to D, unless the two lines have a divergency.

2. Two lines which have two of their corresponding points equidistant, have them all so, and are parallel; because two points determine the position of a straight line (3).

3. Parallel lines can never meet, though produced to an infinite length.

11. THEOREM 1. A line EF (Fig. 3.) intersecting parallel lines, such as AB and CD, makes with them equal *corresponding* angles. Corresponding angles are those which are on the same side of each parallel, and of the intersecting line, such as M and m, P and p, S and s, R and r.

DEM. It is a self-evident thing that the two lines AB and CD having no divergency, their common secant EF must have the same divergency with both.

For perspicuity's sake, suppose the line AB and the secant EF to remain immoveable, and the line CD to move towards AB, without altering its divergency with the secant; when the two intersections coincide, the four angles on each will coincide; and then the line CD coincides also with AB, unless it has lost its parallelism

lelism with AB, which cannot be supposed without altering the angles m, p, s, r .

Moreover, suppose PL to have some divergency with PB; the angle $EPB = EPL + BPL$. If then PL were moving so as to lose its divergency with PB, the angle EPL should be equal to the angle EPB. But CD has no divergency with PB. Therefore the angle $P = p$, and of course $M = m, R = r, S = s$.

The reciprocal of the theorem is true, viz. if a line makes equal corresponding angles with two lines, the two lines are parallel.

12. THEOREM 2. A secant makes with two parallel lines equal *alternate interior* and equal *alternate exterior* angles.

The angles such as S and p, R and m , on opposite sides of the secant, and inside the parallels, are the alternate interior; P and s, M and r are the alternate exterior angles.

DEM. By (9 . 3) $P = S$; and by (11) $P = p$. Then $S = p$; and from the same principle $R = m, P = s, M = r$.

The reciprocal is also true, viz. two lines are parallel, if the same secant makes with them equal alternate interior or exterior angles.

Perpendicular and oblique Lines.

13. DEFINITIONS. A line is *perpendicular* to another line, if it meets it with an equal divergency on both sides. So AP (Fig. 4.) is perpendicular to BD if $APB = APD$.

A line is *oblique* to another which it meets with unequal divergency on both sides. So AE is oblique to BD if $AEB > AED$.

C

COROLL.

14. COROLL. 1. Perpendicular lines make with each other angles of 90° ; and reciprocally.

2. Two lines are reciprocally perpendicular to each other. Since any two contiguous angles are 180° (9. 2) each makes, on the two sides of the other, two angles of 90° each.

3. A line perpendicular to another has every one of its points equidistant from any two points of the second, taken at equal distances from the meeting. So, if $PB=PD$ and $PE=PF$, the two distances from every point of the perpendicular, such as C and A , to E and F , or to B and D , are equal. For if the distances of such points as A and C were less to F than to E , to D than to B , the line AP would evidently diverge less from PD than from PB .

And, as two points fix the position of a line, all the points of AP are known to be equidistant from B and D , if any two, such as A and C or A and P , are ascertained to be so.

Moreover, the perpendicular AP will meet all the points which are equidistant from B and D , if sufficiently produced. Hence,

4. Oblique lines, such as AE and AF , or AB and AD drawn from the same point A of the perpendicular to points on each side of it, equidistant from the meeting P , are equal. For, by the last, these oblique lines measure equal distances. And besides, if the figure be folded on the perpendicular AP , the point E will coincide with F , and B with D ; and, of course, AE will coincide with AF ; and AB with AD .

The same is true of oblique lines drawn from the extremities of equal perpendiculars to equal distances from their meeting. So, in the supposition of the three perpendiculars $AP=RF=LP$, the oblique lines $AF=RP=LF=AE$.

5. A

5. A line which has any two of its points equidistant from the two extremities of another, bisects it perpendicularly. Also, a line perpendicular on another, and bisecting it, passes through all the points equidistant from its extremities.

And a line perpendicular on another, having any one of its points equidistant from its extremities, bisects it. Whence,

6. A radius or diameter perpendicular on a chord, bisects the chord; and a radius bisecting a chord is perpendicular on it. Also, any line bisecting a chord perpendicularly passes through the centre of the circle.

7. Two lines perpendicular to the same line are parallel; and so are two oblique lines of the same divergency. Because the third line makes equal corresponding angles with the two.

15. THEOREM 1. The perpendicular is the shortest line from a point to another line.

DEM. Suppose the perpendicular AP to be produced to L, so that $AP = PL$; and draw from A and L, to any distance from the intersection P, the oblique lines AF and LF. The straight line AL is shorter than the angular line AFL. Therefore the half of AL, which is the perpendicular AP from A to BD, is shorter than the half of AFL, which is the oblique line AF from the same point A to BD.

The same demonstration is applicable to every oblique line from A to BD.

16. COROLL 1. The perpendicular line is the only true measure of distance from a point to a line. Because there is but one perpendicular which is of an invariable length; while the oblique lines are variable and increasing as they remove from the perpendicular, which, of course, is the limits of the decreases of the oblique lines on both sides of it.

2. Per-

2. Perpendicular lines between parallels are equal and parallel. Because they are perpendicular to the same, and measure the equal distances of the corresponding points of the parallels.

The same is true of oblique lines between parallels, if they are of the same divergency with them. Because they are oblique lines from the extremities of equal perpendiculars to the same distance from the meetings of the perpendiculars.

17. THEOREM 2. A radius to the point of contact is perpendicular to the tangent. Reciprocally, a line perpendicular to the tangent at the point of contact passes through the centre of the circle.

DEM. The radius at the point of contact is the shortest line from the centre to the tangent, since any other line from the centre meets the tangent out of the circle.

18. COROLL. 1. A straight line perpendicular to the extremity of a radius is a tangent to the circle.

2. The same line TS (Fig. 2.) may be a tangent common to an infinite number of circles of different magnitudes, which have their centres on the diameter AB or its prolongation.

19. THEOR. 3. A radius perpendicular to a chord bisects the arc sustained by the chord.

DEM. The radius CA (Fig. 2.) perpendicular on the chord LN bisects it, (14. 6.) and has, of course, the point A equidistant from the extremities of the chord LN. Therefore the two chords and arcs $AL = AN$. (9. 6.)

COROLL. Two arcs of the same circle contained between parallel lines, are equal.

DEM. Suppose LN and MD to be parallel chords, and the radius CA perpendicular on them. Then, by the last, $AL = AN$, and $AD = AM$. Whence $AM - AL = ML = AD - AN = DN$.

From

From the above principles may be resolved the following problems of practical Geometry :

20. **PROBLEM 1.** To find a point equidistant from the two extremities, or from any other two points of a given line ?

SOLUTION. From the two extremities of the line as centres, and with the same opening of the compasses, describe two arcs intersecting each other. Their intersection is equidistant from the two extremities of the line; since the two distances are two radii of equal circles.

21. **PROBLEM 2.** To draw a line perpendicular on a given line ?

A perpendicular may be required either to or from an indeterminate point, to or from a given point of the line. There are four cases.

SOL. 1. To an indeterminate point of a given line ?

Find out two points equidistant from any two points of the given line. Draw a straight line through these two points; it will be perpendicular to the given line. (14. 5.)

2. To a given point of a line ?

Take with the compasses, on that line, two points equidistant from the given point; and find, out of the line, another point equidistant from the two. The line joining this and the given point is the perpendicular required.

3. From a given point out of a line ?

From that given point describe a circle intersecting the line at two points; and find another point equidistant from the two intersections. The line joining this and the given point is a perpendicular to the line.

4. To

4. To the extremity of a line?

Produce the line, if possible, and the solution is the same as the second.

If the line could not be produced, erect a perpendicular on any point of it; and draw by its extremity a line parallel to that perpendicular, by the method hereafter (23.) this will be the perpendicular required.

22. PROBLEM 3. To draw a tangent to a given point of the circumference of a circle?

SOL. Draw a radius to the given point, and a line perpendicular to the extremity of the radius. (18.)

23. PROBLEM 4. To draw a line parallel to another given line?

SOL. Erect two perpendiculars on the given line, and draw a line cutting equal parts of the two perpendiculars (16. 2.) . . . , or from the two extremities, or any other two points of the given line, describe two equal circles; and draw a line tangent to both; it will be the parallel required.

24. PROBLEM 5. To find the centre of a given circle?

SOL. Draw any two chords, and bisect them by perpendiculars; the two perpendiculars intersect each other at the centre of the circle. (14. 6.)

25. PROBLEM 6. To describe the circumference of a circle through any three given points, which are not in the same straight line?

SOL. Join the three given points by two straight lines, and bisect them by two perpendiculars; their intersection is the centre of the circle required. (24.)

As all the above problems are immediate inferences from the foregoing principles, it will be sufficient to exhibit

exhibit in the explanation the figures appropriate to the solutions.

Of Angles, and their Measure.

26. DEFINITION. An angle is the divergency or inclination of two lines meeting at one of their extremities, as above. (6.)

The two lines are said to be *diverging* from, or *converging* towards their meeting. They are called the *sides*, and their meeting point the *vertex* of the angle. The sides are considered as indefinite; so that their being longer or shorter does not change the magnitude of the angle, as it does not change their divergency. An angle is commonly designed by the single letter at the vertex, or by three letters, one at the vertex, and one at each side; the one at the vertex being placed between the other two. An angle may increase or decrease, and of course is, like any other quantity, susceptible of measurement.

27. It has been already demonstrated (7.) that the proper measure of a *central* angle (which has its vertex at the centre) is the arc of the circumference contained by its two sides. A graduated circle is the instrument used in practical geometry for the measurement of central angles.

Angles take different names on account of their magnitude. A *right* angle is that which is measured by an arc of 90° , the fourth part of the circumference. An *acute* angle is measured by less than 90° . An *obtuse* angle is measured by more than 90° . What an acute angle or arc wants of 90° , is called its *complement*; and what an angle or arc wants of 180° , is called its *supplement*. Angles which have the same or equal complement

complement or supplement are equal, and reciprocally. The right angle has no complement, and is equal to its supplement.

Angles take also different names according to the situation of their vertex with respect to the circle. Their measure is derived from that of the central angle, and will be given with their names, after premising some general properties of angles, most of which were already demonstrated, and are collected under the following number :

28. Perpendicular lines make right angles (14). A radius to the point of contact makes right angles with the tangent ; and so does a radius which bisects a chord. (17 and 19.)

The sum of all the angles about the same point is four right angles, or 360° . The whole circumference of a circle is the sum of all their measures. (6.)

Any two lines intersecting each other make four angles, the sum of which is four right angles, or 360° . And the sum of any two contiguous angles is two right angles, or 180° . (9. 2.)

Two lines intersecting each other make equal vertical angles. (9. 3.)

A straight line intersecting parallel lines makes with them equal corresponding, equal alternate interior, and equal alternate exterior angles. (11 and 12.)

29. THEOREM 4. The measure of an angle of the *segment* (formed by a tangent and a chord) is half the arc sustained by the chord.

DEM. The chord AB (Fig. 5.) makes with the tangent TN two angles of the segment ; one BAT, measured by half the arc AB, the other BAN, measured by half the arc ALB. Draw the radius CA to the point of contact, and the two diameters FE parallel, and LP perpendicular to the chord AB ; then $CAT = FCP$ two right angles (28.) and $CAB = FCA$

two

two alternate interior angles. Whence, the angle of the segment $BAT = CAT - CAB = ACP = FCP - FCA$. But the measure of the central angle ACP is $AP = \frac{1}{2}AB$ (19). Therefore the angle of the segment BAT , which is equal to the central angle, has also for its measure $\frac{1}{2}AB$.

The other angle of the segment BAN and the central angle ACL have then equal supplements, BAT and ACP , and, of course, are equal (27) and are measured by the arc $AL = \frac{1}{2}ALB$. Besides $BAN = CAN + CAB = LCF + FCA = LCA$ measured by AL .

30. THEOREM 2. The measure of an *inscribed* angle (formed by two chords) is half the arc contained by its sides.

DEM. The inscribed angle BAD (Fig. 5.) is equal to the angle of the segment DAT less the angle of the segment BAT ; and, of course, its measure is the half of the arc DBA less the half of the arc BA (29) which is the half of the arc BD .

31. COROLL 1. An angle of the segment and an inscribed angle are equal, if the arc sustained by the chord of the first is the same or equal to the arc contained by the sides of the other.

2. A central angle is double of an inscribed angle containing the same or an equal arc.

3. An inscribed angle is right, acute, or obtuse, as the arc contained by the sides is equal to, less or greater than the half of the circumference.

4. Two chords from the extremities of the same diameter and meeting at any point of the circumference make an inscribed right angle, and are perpendicular to each other.

32. THEOREM 3. The measure of an *eccentric* angle (having its vertex between the centre and the circumference) is half the sum of the two arcs contained by the sides and by their prolongation.

D

DEM.

DEM. Produce to L and E (Fig. 6.) the two sides of the eccentric angle BAD; and from either L or E draw the chord EF parallel to one of the sides AB. Then the eccentric angle BAD is equal to its corresponding inscribed angle FED (11) and its measure is $\frac{1}{2}DF$ (30) $= \frac{1}{2}DB + \frac{1}{2}BF = \frac{1}{2}DB + \frac{1}{2}EL$ (20).

33. COROLL. The eccentric angle decreases as its vertex removes from the centre; because EL decreases, BD remaining constant. Hence an angle decreases as its sides become longer from the same two extreme points, and reciprocally.

34. THEOREM 4. The measure of a *circumscribed* angle (formed out of the circle by two secants) is half the difference of the two concave and convex arcs contained by its sides.

DEM. Let BAD (Fig. 7.) be the circumscribed angle to be measured. From the intersection of either of the two secants with the circumference draw the chord LF parallel to the other secant AB. Then the circumscribed angle BAD is equal to its corresponding inscribed angle FLD, and has the same measure, which is $\frac{1}{2}FD = \frac{1}{2}BD - \frac{1}{2}BF = \frac{1}{2}BD - \frac{1}{2}EL$.

35. COROLL. The circumscribed angle BAT, made by a secant and a tangent, is also measured by half the difference of the concave and convex arcs.

DEM. Draw the chord EG parallel to the tangent, and the radius CT to the point of contact, which bisects the arc EG at T (19), then the angles $BAT = BEG$, and have the same measure $\frac{1}{2}BG = \frac{1}{2}BT - \frac{1}{2}GT = \frac{1}{2}BT - \frac{1}{2}ET$.

Practical Geometry resolves the following Problems from the above principles :

36. PROBLEM 1. To bisect a given angle ?

SOL. From the vertex of the given angle, as a centre, describe a circle intersecting the two sides; join the two intersections by a chord; and a line from the

the vertex perpendicular to the chord will bisect the chord, the arc it sustains, and the given angle measured by the arc (19).

37. PROBLEM 2. To double a given angle?

SOL. From the vertex, as a centre, describe a circle intersecting the two sides; from either of the intersections draw a chord perpendicular on the other side; and from the centre, a radius to the extremity of the perpendicular chord. The radii at the two extremities of the perpendicular chord are the double angle (31. 2).

Again. Describe a circumference passing at the vertex of the given angle and intersecting the two sides. Two radii from the centre to the two intersections make a central angle double of the given one.

38. PROBLEM 3. To make at any point of a given line an angle equal to another given angle?

SOL. From the vertex of the given angle and from the given point on the line describe two equal circles; and from the intersection of the circle with the given line measure an arc equal to the arc of the other circle contained by the sides of the given angle. The radius to the extremity of that arc makes with the given line the angle required.

OF FIGURES.

39. STRAIGHT lines enclosing a space constitute a *figure*, which has as many angles as sides.

Three lines at least are requisite to enclose a space, and make the most simple of all figures, called *triangle*. A figure of four sides takes the name of *quadrilatera* or *quadrangle*; of five sides, is called *pentagon*; of six sides, *hexagon*; of seven sides, *heptagon*; of eight sides, *octagon*, &c.; and their common name is *polygon*; which again takes different names, on account of the relative magnitude of the angles and situation of the sides, to be mentioned hereafter.

As all figures may be decomposed into triangles, the geometrical properties of polygons are mostly inferred from the properties of triangles. Therefore, we have to consider under the present head: 1st. The triangle and its general properties. 2. The equal triangles. 3. The quadrilatera. 4. The polygons in general. 5. The proportional lines. 6. The similar triangles and polygons. 7. The proportional lines with regard to similar triangles, to the circle, to similar polygons.

Of the Triangle and its general Properties.

40. A TRIANGLE is a figure of three angles made by the meeting of three straight lines, called its sides. It is obvious that any of the three sides is shorter than the sum of the other two.

A triangle takes different names on account of its sides and angles. On account of its sides, it is called
equilateral,

equilateral, when the three sides are equal; *isosceles*, when two of the sides are equal; *scalene*, when the three sides are unequal. On account of the angles, it takes the name of a *right-angled* triangle, when it has a right-angle; of *acute-angled* triangle, when its three angles are acute; of *obtuse-angled* triangle, when it has an obtuse angle.

41. **THEOREM 1.** The sum of the three angles of every triangle is equal to the sum of two right-angles, or 180° .

DEM. Every triangle may be inscribed in a circle (its circumference passing at the vertex of each of the three angles) since the three angles are three points which cannot be in the same straight line (25) then the three angles are inscribed, and contain the entire circumference; each of them is measured by the half of the arc it contains (30). Therefore, the sum of the measures of the three angles is 180° , the half of the circumference, the same as the measure of two right angles.

42. **COROLL. 1.** A triangle cannot have more than one right angle or one obtuse angle.

2. Any angle of a triangle is supplement to the sum of the other two angles; and reciprocally.

3. Two triangles which have two equal angles, or the sum of two of their angles equal, have the third angle equal.

4. In a right-angled triangle either of the two acute angles is complement to the other.

5. Two angles which have their two sides perpendicular each to each are equal.

DEM. Suppose EB (Fig. 8.) perpendicular to AD, and EF perpendicular to AB; then the two triangles EDS, AFO are right-angled, and have each an equal acute angle at S and O (9. 3) whence the other acute

acute angles E and A, made by the perpendiculars, are equal.

6. If a right-angled triangle be inscribed in a circle, the side opposite to the right angle, called *hypotenuse*, is the diameter of the circle (31. 4).

43. THEOREM 2. In every triangle equal sides are opposite to equal angles; the greatest side, to the greatest angle; the smallest side, to the smallest angle; and reciprocally.

DEM. A triangle being inscribed in a circle, every side is the chord of the arc, the half of which measures the opposite angle (30), equal chords sustain equal arcs, which measure equal angles; the greatest chord sustains the greatest arc, which measures the greatest angle; and the smallest chord sustains the smallest arc, which measures the smallest angle; and reciprocally (9. 6). Therefore, &c.

44. COROLL. 1. An equilateral triangle is also equiangular: an isosceles triangle has two equal angles opposite the two equal sides: a scalene triangle has its three angles unequal; and reciprocally.

2. An inscribed equilateral triangle trisects the circumference. An isosceles triangle bisects the segment containing the two equal sides.

45. THEOREM 3. A line from the vertex of the unequal angle of an isosceles triangle perpendicular on the opposite side, bisects both the angle and the side.

DEM. As in the isosceles triangle BAD (Fig. 9.) the sides $AB = AD$, the line APR, from the vertex A, perpendicular on the opposite side BD, has all its points equidistant from B and D (14. 5). Therefore, $BP = PD$ and the side BD is bisected. Also, the two chords and arcs $RB = RD$; and the two angles $BAR = DAR$ (30) and the angle BAD is bisected.

46. COROLL.

46. COROL. A perpendicular from any angle of an equilateral triangle to the opposite side, bisects both the angle and the side. Because every one of the three angles may be taken for the vertex of an isosceles triangle.

47. THEOREM 4. The chord of 60° is equal to the radius of the circle.

DEM. Let ABD (Fig. 10.) be an equilateral inscribed triangle. The arc BD is one-third part of the circumference (44. 2). The line AR perpendicular to BD bisects its arc at R (46); therefore the arc BR is the one-sixth part of the circumference; and its chord BR is a chord of 60° . From the centre which is in the line AR (14. 5) draw the radius CB. Then the central angle BCR is of 60° , and as in the triangle BCR the two sides $BC = CR$, so their opposite angles $CRB = CBR$ (43); and each is half the supplement of the central angle BCR (42. 2), which half supplement is 60° . Hence, the triangle BCR is equiangular, and, of course, equilateral; and the side BR the chord of 60° equals the radius CR.

48. COROL. Six chords equal to the radius will sustain exactly the entire circumference, and make an inscribed regular hexagon. Each of the six sides of the hexagon is shorter than the arc it sustains; therefore the circumference of a circle is greater than six radii, or three diameters. The exact ratio of the radius to the circumference is the celebrated problem of the *Quadrature of the Circle*; it remains incommensurable to this day, notwithstanding the labours of the most able geometers. One of the simplest methods of resolving the problem by approximation shall be inserted hereafter in this treatise.

49. THEOREM 5. If any of the three sides of a triangle be produced, the exterior angle is equal

equal to the sum of the two interior opposite angles.

DEM. The exterior angle ADF (Fig. 11.) is the supplement of the interior contiguous angle ADB (28); the same angle ADB is supplement to the sum of the other two interior angles $A + B$ (42. 2). Therefore, the exterior angle $ADF = A + B$ the sum of the two interior opposite angles.

50. COROL. If the three sides of a triangle be produced, the sum of the three exterior angles is four right angles, or 360° .

DEM. The sum of every exterior and contiguous interior angle is 180° . Then the sum of the three exterior and the three interior angles is $3 \times 180^\circ$; from which subtracting the sum of the three interior ones, which is 180° , there remain for the sum of the three exterior angles 360° .

Equal Triangles.

51. Two triangles, or any two figures of the same number of sides, are *equal*, when the sides and angles of one are equal to the sides and angles of the other, each to each; so that one being superimposed on the other, all the sides and angles of both would coincide two and two. The sides opposite equal angles in different triangles or figures are called *homologous* sides. Equal triangles are of great use in all branches of geometry; therefore, it is important to know all possible ways of ascertaining their equality: it is ascertained by any three equal things, except the three angles, which comprehends the five following cases:

52. THEOREM 1. Two triangles are equal which have an equal angle contained by two equal sides.

DEM.

DEM. Let (Fig. 12,) the two angles $A \equiv a$, and the sides $AB \equiv ab$ and $AD \equiv ad$. Superimpose the angle a on the angle A ; they will coincide as well as the sides ab with AB and ad with AD ; of course, the third side bd will coincide with BD ; and the two triangles are equal.

53. THEOREM 2. Two triangles are equal, which have an equal side between two equal angles.

DEM. Let $BD \equiv bd$, $B \equiv b$, $D \equiv d$. The side BD being superimposed on bd will coincide with it; and so will the angles B with b , and D with d ; which evidently cannot be so, unless the two other sides coincide, AB with ab , and AD with ad .

54. THEOREM 3. Two triangles are equal which have two equal angles and an equal side each opposite either of the two equal angles.

DEM. Let $B \equiv b$, $D \equiv d$, $AB \equiv ab$. Superimpose ab on AB ; they will coincide as well as b with B ; then, all things coincide, provided ad coincides with AD ; which is necessarily so; because the side ad originating at A , like AD , will coincide with AD , and meet BD at D , unless it meets it inside or outside the triangle ABD ; which two are equally impossible, in the supposition of the angles $D \equiv d$ (33).

55. THEOREM 4. Two triangles are equal which have two equal sides and an equal angle each opposite either of the two equal sides; provided the angle opposite the other equal side in each is of the same kind.

DEM. Let $AB \equiv ab$, $AD \equiv ad$, $B \equiv b$. Superimpose ab on AB ; they will coincide, as well as the angles b with B . Draw now the perpendicular AP , take $PF \equiv PD$, and join AF to have $AD \equiv AF$ (14. 4). The side ad cannot meet the side BD and remain equal to A unless they are both perpendicular to BD and coincide

E

with

with AP, or both oblique to BD, and either coincide with AD, or one with AD and the other with AF. In the two first cases the two triangles are equal; they are unequal in the third; but the angle D of one is acute, and AFB of the other is obtuse, consequently not of the same kind. Therefore if D and d are of the same kind, the two triangles are equal.

So this supposition leaves a dubious case, which, however, is seldom liable to error; as the state of the question generally determines whether the angles d and D are of the same kind or not.

56. THEOREM 5. Two triangles are equal which have their three sides equal, each to each.

DEM. Let $AB=ab$, $AD=ad$, $BD=bd$. Superimpose any two equal sides, bd on BD ; they will coincide. So shall the two angles at their extremities coincide, b with B, d with D: for if b were greater or less than B, the opposite side ad should be longer or shorter than AD, contrary to the supposition. Then $b=B$, and the two triangles are equal (52).

57. THEOREM 6. Two triangles may have their three angles equal, two and two, without being equal.

DEM. Draw EG parallel to bd , and produce to it ab and ad . The two triangles bad , GaE , are not equal, though they have their three angles equal, viz. an angle common to both, $b=G$ and $d=E$ corresponding angles. Such triangles are called *similar triangles*; of which hereafter.

58. General THEOREM. Two triangles are equal which have any three things equal, among which at least one side. It follows from the above theorems. But only two equal things always leave the equality of triangles dubious, as is obvious.

59. COROLL

59. COROLL. 1. Two right-angled triangles are equal, which have, besides the right angle, any two things equal, one of which is a side.

2. A radius perpendicular to a chord bisects both the chord and the arc it sustains. The same as (19).

DEM. Let the radius CR (Fig. 13.) be perpendicular to the chord AB; and join their extremities by the radii CA, CB, and the chords AR, BR. The two right-angled triangles $CPA = CPB$, on account of the common side CP and $CA = CB$. Therefore the third side $AP = BP$. Likewise the two right-angled triangles $APR = BPR$, on account of the common side PR and $AP = BP$: of course, the two chords $AR = BR$, and sustain equal arcs (9. 6).

3. Two chords AB, DE equidistant from the centre of the circle are equal; and reciprocally.

DEM. From the supposition, the perpendiculars $CP = CK$ (16); therefore the right-angled triangles $ACP = DCK$. Whence the half chords $AP = DK$; and the entire chords $AB = DE$.

The Quadrilateres.

60. A QUADRILATERE, a figure of four sides, is called *square*, when its four sides are equal and perpendicular, making four right angles. They call it *parallelogram*, if only the opposite sides are equal and parallel, two and two; *rectangle*, if the four angles of the parallelogram are right angles; *rhomboid*, if the four sides are equal, and making two opposite angles acute, and two opposite ones obtuse; *trapezium*, if it has only two opposite sides parallel. A *diagonal* is a straight line joining two opposite angles of a figure.

61. THEOREM

61. **THEOREM 1.** A diagonal divides a parallelogram into two equal triangles.

DEM. The parallelogram $ABED$ (Fig. 14.) has the side AB parallel to DE , and AD parallel to BE . Therefore the two triangles ABD and BED have a common side, the diagonal BD , and the alternate interior angles $BDA = DBE$ and $ABD = BDE$; wherefore they are equal (58).

62. **COROLL. 1.** A triangle may always be completed into a parallelogram double of itself, by drawing from any of the three angles a line equal and parallel to the opposite side, and joining its extremity with the nearest of the other two angles. So the triangle BDE being given, from the angle B draw BA equal and parallel to DE , and join A and the angle D .

2. Parallel lines between parallels are equal; so are also perpendicular lines. The same as (16. 2).

DEM. The lines AB and DE , AD and BE , being supposed parallel, the diagonal BD makes two equal triangles ABD and BDE , in which the homologous sides $AB = DE$ and $AD = BE$; and if you draw the two perpendiculars BP to DE and DK to AB , the two triangles $BPD = BKD$; and their homologous sides $BP = DK$.

63. **THEOREM 2.** The sum of the four angles of any quadrilatre is constantly four right angles, or 360° .

DEM. A diagonal divides any quadrilatre into two triangles; and the sum of the four angles of the quadrilatre is the same as the sum of the six angles of the two triangles, viz. twice two right angles, or 360° . (41.)

64. **COROLL. 1.** If the four sides of a quadrilatre be produced, the sum of the four exterior angles will be constantly four right angles, or 360° .

DEM.

DEM. The sum of every exterior and its contiguous interior angle is two right angles (9. 2.) ; then, the sum of the four exterior and the four interior angles is eight right angles ; from which subtracting four right angles, the sum of the four interior angles ; remain four right angles for the sum of the four exterior ones.

2. The sum of any two opposite angles of a quadrilatre inscribed in a circle, is constantly two right angles.

DEM. The quadrilatre ABDF (Fig. 15.) being inscribed, the angle B is measured by half the arc AFD ; and the opposite angle F, by half the arc ABD (30) : the sum of their measures is then half the circumference, the same as of two right angles. The same for the sum of the other two opposite angles A and D. It would not be so, if the quadrilatre was not inscribed ; for if the angle B, for instance, was inside or outside the circumference of the circle, its measure would be greater or less than half the arc AFD. (32. 34.)

65. **THEOREM 3.** The sum of two contiguous interior angles of a parallelogram is two right angles ; and, of course, the two opposite angles are always equal.

DEM. The side AB (Fig. 14.) being produced to F, the sum of the interior angle ABE and the exterior one EBF is two right angles (9. 2.) ; and the interior angle A contiguous to ABE is corresponding and equal to the exterior one EBF. The same may be said of any other two contiguous interior angles. Whence, either of the two opposite angles, such as A and E, added to ABE makes with it the same sum, and, of course, are equal.

Of.

Of Polygons.

66. **EVERY** figure is a polygon. It is called *regular* or *irregular*, according as all its sides and angles are equal or unequal; and *symmetric* when its opposite sides are equal and parallel, two and two. A polygon is *inscribed* in a circle, the circumference of which passes through the vertex of every angle; it is *circumscribed*, when all its sides are tangents to a circle, which is then inscribed in the polygon.

67. **THEOREM 1.** The sum of all the interior angles of a polygon is equal to as many times two right angles, or 180° , as there are sides less two sides.

DEM. From any angle A (Fig. 16.) of a polygon such as ABDEF, as many diagonals like AD and AE may be drawn to the opposite angles as there are sides less two, making within the polygon as many triangles as sides less two, such as ABD, ADE, AFE; and the sum of all the angles of the triangles is obviously the same as of all the angles of the polygon. That is two right angles, or 180° , as many times as sides, less two sides. (41.)

68. **COROLL. 1.** All the sides of a polygon being produced, the sum of all the exterior angles is constantly four right angles, or 360° , whatever be the number of sides.

DEM. The sum of every exterior and its contiguous interior angle is two right angles. Taking then n for the number of sides, r for one right angle, the sum of all the exterior and interior angles of any polygon whatsoever is $2nr$. But the sum of all the interior
angles

angles is $n-2 \times 2r = 2nr - 4r$ (67.), which being subtracted from $2nr$ leaves for the sum of all the exterior angles $2nr - 2nr + 4r = 4r = 360^\circ$.

2. As the angles of a regular polygon are all equal, the magnitude or measure of the common angle of a regular polygon will be $a = \frac{n-2 \times 2r}{n}$. That formula gives for the common angle of an equilateral triangle $\frac{3-2 \times 180^\circ}{3} = 60^\circ$. For the common angle of a regular pentagon $\frac{5-2 \times 180^\circ}{5} = 108^\circ$. For the common angle of a regular hexagon $\frac{6-2 \times 180^\circ}{6} = 120^\circ$; for a regular octagon 135° , &c.

69. THEOREM 2. A regular polygon may be inscribed in, and circumscribed about a circle.

DEM. Suppose ABDEFG (Fig. 17.) to be a regular polygon: it has all its sides and angles equal. (66.) Bisect the angles by straight lines, such as AC, BC, GC, &c. They will make as many equal triangles as there are sides, since they have an equal side each, viz. the side of the polygon; and two equal angles, the halves of the angles of the polygon (58.); which requires all the lines bisecting the angles to meet at the same point C, and be equal. Then, a circle described from C as a centre, with the radius CA, will pass by the vertex of every angle of the polygon; and the polygon be inscribed in that circle. These radii bisecting the angles of the polygon are called *oblique radii*.

Now, from the centre C draw perpendiculars to the sides of the polygon, such as CP, CK, &c. They will bisect the sides (45.) and make equal triangles with the oblique radii and the half sides of the polygon $CPA = CKA = CPB$, &c.; whence $CP = CK$, &c. And

And if a circle be described with the radius CP , all the sides of the polygon will be tangents to it (18.), and the polygon be circumscribed about the circle. Such radii are called *right radii*.

70. COROL. The chord of 60° , which is the side of the regular inscribed hexagon, is equal to the radius of the circle. As (47).

DEM. In a regular inscribed hexagon the triangles, such as ACB , made by each side, with two oblique radii are equiangular; the central angle being on an arc of 60° , and each of the other two angles, the half of the common angle of the hexagon. Whence the triangles are equilateral, and $AB=AC$.

71. THEOREM 4. The oblique radius of an equilateral triangle is double of the right radius.

DEM. The equilateral triangle BEG (Fig. 17.) being inscribed in the circle, its right radius CL produced to A bisects the chord and arc BG ; and is bisected by it, on account of $BC=CG=BA=GA$ (14. 5). Therefore the oblique radius $CG=CA=2CL$.

72. COROLL. 1. In a right-angled triangle which has one of its acute angles of 60° , the other of 30° , the side opposite to the acute angle of 30° is half the *hypotenuse*. The triangle CLG is the supposed case; and its hypotenuse $CG=2CL$ the side opposite to the acute angle of 30° .

2. The side of a regular inscribed polygon under 6 sides is longer than the radius of the circumscribed circle: it is shorter in a regular polygon above 6 sides.

3. As the number of the sides of a regular inscribed polygon increases, the right radius increases towards an equality with the oblique radius; because the sides, becoming shorter chords, remove from the centre.

centre. But the right radius cannot reach the equality with the oblique one until the number of the sides of the polygon becomes infinite; in which case, the sides of the polygon are infinitely small, coinciding sensibly with the arcs they sustain, and the regular polygon is sensibly the circle itself.

4. There are but three regular polygons, the angles of which joined together fill exactly a space, viz. six equilateral triangles, four squares, three regular hexagons, the sum of which is 360° . But a space may be filled by the angles of some regular polygons of a different number of sides, such as two equilateral triangles and two regular hexagons. A space may also be filled by innumerable irregular ones.

Proportional Lines.

PROPORTIONAL lines are lines in a geometrical proportion. Lines being measurable quantities, the theory of proportions is applicable to lines no less than to arithmetical quantities.

73. THEOREM 1. Lines drawn from any point of one of the three sides of a triangle, parallel to the two other sides, divide the three sides into proportional parts, which are also proportional to the entire sides.

DEM. Let the side AB of the triangle ABD (Fig. 18.) be divided into any number of equal parts at E, F, G, and draw from the divisions the lines EP, FR, GT parallel to AD, and EL, FM, GN parallel to BD; then, LT, MR, NP parallel to AB. Such parallel lines will make on each of the three sides of the triangle ABD the same number of triangles; such as AEL, MSN, SRP, &c. all equal to one another,

F

on

account of all their homologous sides being parallel between parallels, and their homologous angles corresponding. Therefore, in the supposition of $AE=EF=FG=GB$, the other two sides AD and BD are likewise divided into equal parts $AL=LM=MN=ND$ and $BT=TR=RP=PD$, which are homologous sides of the triangles. Therefore, they are the same aliquot parts of the sides; and, of course, proportional to themselves and to the entire sides. Thus, $AE:AB::AL:AD::BT:BD$. Also, $AE:AG::AL:AN::BT:BP$; Also, $AE:EG::AL:LN::BT:TP$. Likewise, $AE:EL=BT::AF:FM=BR$. And $BT:BP::TG=DN:PE=DL$, &c. &c.

Now, had the side AB been divided into 10, or 100, or an infinite number of equal parts, at every one of its elementary points; lines from the divisions parallel to the two other sides would have made on each side 10, or 100, or an infinite number of equal triangles, and divided the sides into as many proportional parts. Taking then such parallel lines from any point C of AB , such as CH, CX, HV , we have the proportion $AC:AB::AH:AD::BV=CH:BD$, &c.

74. PROBLEM. To find a line proportional to three other given lines?

SOL. Let the three given lines be A, B, D . Take any two of them and place them in a straight line, for instance B and D making the straight line AD , and meeting at H ; so that $B=AH$, and $D=HD$. With AD and the third line A make any angle such as A , and suppose the line to terminate at C ; so that the three given lines $B=AH, D=HD, A=AC$. Join C and H , and from D draw DB to AC produced parallel to CH . Then CB is the fourth line proportional to the three given lines; since, from the above, $B=AH:A=AC::D=HD:CB$.

75. THEOREM

75. THEOREM 2. The homologous sides of two unequal equiangular triangles are proportional.

DEM. Let the two triangles ABD and abd (Fig. 19.) be equiangular, $A=a$, $B=b$, $D=d$. Take $AF=ab$, and draw FE parallel to BD . Then the triangle AFE is equal to the triangle abd , on account of $A=a$, $B=b=AFE$, and $AF=ab$ by supposition (53.); whence $AE=ad$ and $FE=bd$. But because FE is parallel to BD , we have $AF=ab : AB :: AE=ad : AD :: FE=bd : BD$ (73.); or $ab : AB :: ad : AD :: bd : BD$, the proportion of the homologous sides.

The proportional lines are of the most universal use in all the branches of geometry and physics. Therefore, the last theorem shall be henceforth the most universal principle. And as its application requires equiangular, also called *similar* triangles, it is of great moment to know all the means of ascertaining the similarity of triangles.

Similar Triangles.

76. DEFINITION. Two triangles are *similar* which have their angles equal, each to each. Equal triangles are similar; but they may be similar without being equal (57.). The equal angles of similar triangles are their *homologous* angles; and their sides opposite to the equal angles are their *homologous* sides.

77. THEOREM 1. Two triangles are similar which have two angles equal each to each.

DEM. Any angle of a triangle is supplement to the other two, because the sum of the three is constantly 180° (41.). Therefore, two triangles, which
have

two angles equal each to each, have their three angles equal, and are similar.

78. COROL. Two triangles are similar when they have an equal angle each, besides a common angle. Also, when they have a common angle at the vertex and parallel bases.

79. THEOREM 2. Two triangles are similar which have their homologous sides parallel.

DEM. Suppose AB parallel to ab , AD to ad , and BD to bd . Produce one of the three sides, for instance ad , so as to meet the side BD, either inside or outside the other triangle ABD. Then, the corresponding angles $A = BRG = a$, and $D = BGR = d$. Then also $B = b$.

80. THEOREM 3. Two triangles are similar which have their homologous sides perpendicular.

DEM. Two sides of one make the same angle as the two sides of the other to which they are perpendicular (42. 5.). Therefore, the three sides of one, being perpendicular to the three sides of the other, make their three angles equal.

81. THEOREM 4. Two triangles are similar which have an equal angle each contained by proportional sides.

DEM. Suppose the two angles $A = a$, and the proportion $AB : ab :: AD : ad$. Take $AF = ab$ and draw FE parallel to BD. Then, $AB : AF = ab :: AD : AE$ (73.). The three first terms of the two proportions are equal. Therefore, the last terms $AE = ad$; and the two triangles $AFE = abd$ (52.); and ABD similar to AFE (78.) is also similar to abd .

82. THEOREM 5. Two triangles are similar which have two proportional sides, and an equal angle each opposite to one of the homologous proportional sides;

sides; provided the angle opposite to the other proportional side be of the same kind in both triangles.

DEM. Suppose the two angles $A=a$ and the proportion $AD : ad :: BD : bd$. Take $AE=ad$; the two angles a and A , if superimposed, will coincide, as well as ad with AE . Draw EF parallel to DB and you have the proportion $AD : AE=ad :: DB : EF$. The three first terms of the two proportions being the same, the last terms must be equal $EF=bd$. But, if you draw EP perpendicular to AB and take $PL=PF$, to have the oblique lines $EF=EL$ (14. 4.), the side db originating at E may coincide with EL as well as with EF . If it coincides with EF , the angle b coincides with the angle AFE and the triangle ADB similar to AEF is similar to adb . If it coincides with EL , the triangles are not similar; but then, the angle B being acute, the angle b , which should be ALF , is obtuse. This dubious case, the same as (55.), is likewise commonly resolved by the state of the question.

83. THEOREM 6. Two triangles are similar which have their three homologous sides proportional.

DEM. Suppose $AB : ab :: AD : ad :: BD : bd$. Take on AB a part $AF=ab$, and draw FE parallel to BD . Then $AB : AF=ab :: AD : AE :: BD : FE$. These three equal ratios have by supposition the same three antecedents, besides the first consequent equal: therefore the other two consequents must be equal $AE=ad$ and $FE=bd$. Of course, the triangles $abd=AFE$ similar to ABD .

The above six theorems are the only six cases in which two triangles are ascertained to be similar to each other. From the proportionality of the homologous sides of similar triangles innumerable properties are

are demonstrated of triangles, polygons and circles : Those shall be selected which may be of use in the various branches of geometry and physics.

Properties of Triangles, from Proportional Lines.

84. THEOREM 1. A straight line, which bisects any angle of a triangle, divides the opposite side into two such segments as to have the following proportion. The opposite side is to the sum of the other two sides as either of the two segments is to its contiguous side.

DEM. Let AF (Fig. 20.) bisect the angle A of the triangle BAD. Draw from either of the other two angles, D for instance, DE parallel to AF, and produce BA to meet that parallel at E. Then BAF and BED are similar triangles (78.); and the triangle DAE is isosceles, on account of the angle $E = BAF = FAD = ADE$; and the two sides $AD = AE$ (44.). Therefore, $BD : BE = BA + AD :: BF : BA :: FD : AE = AD$ (73.).

85. COROLL. Any two sides of a triangle being produced, the parallel bases opposite to the angle contained by the two sides, the perpendiculars from the same angle to the parallel bases, and the sides produced increase in the same proportion.

DEM. Suppose a triangle ABF (Fig. 20.); produce the two sides BA and BF containing the angle B to EP parallel to AF, and draw BP perpendicular to the parallels AF and EP. Then you have the similar triangles ABF to EBD, ABK to EBP, FBK to DBP (78.);

(78.); whence $BA : BE :: BF : BD :: BK : BP :: AE : FD$, &c. &c.

86. THEOREM 2. In an isosceles triangle, which has each of the two angles on the base double of the angle at the vertex, a line bisecting either of the angles on the base divides the opposite side into *middle* and *extreme* ratio: that is, the great segment is middle proportional between the entire side and the small segment.

DEM. Let the isosceles triangle BAD (Fig. 21.) have the angles $B=D=2A$; and DF bisect the angle D, so that $E=O=A$; whence the angle $BFD=O+A=B$ (49.); and the sides $BD=DF=AF$ (43.); and the two triangles BAD, BDF are similar (77.). Therefore, $BF : BD=AF :: BD : AB$, or $\therefore BF : AF : AB$.

87. THEOREM 3. A line from the vertex of a right-angled triangle perpendicular to the hypotenuse divides the right-angled triangle into two small ones similar to the great one and to each other.

DEM. The line AP (Fig. 22.) from the right angle A of the triangle BAD perpendicular on the hypotenuse BD divides the triangle into two right-angled triangles BPA, DPA, each of which has a right angle at P, and an angle common with the great triangle, one at B, the other at D. Therefore the three triangles are similar (77.)

88. COROLL. The perpendicular from the right angle is middle proportional between the two segments of the hypotenuse; and each of the small sides is middle proportional between the hypotenuse and the contiguous segment.

DEM. The two similar triangles DPA, BPA give $\therefore DP : AP : BP$; the two DPA, DAB give $\therefore DP : DA : DB$; and the two BPA, BAD give $\therefore BP : AB : BD$.

89. THE-

89. THEOREM 4. In a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the two small sides.

DEM. The second and third continued proportions of the last give $DP \times BD = AD^2$ and $BP \times BD = AB^2$, whence $\overline{DP + BP} \times BD = BD \times BD = BD^2 = AD^2 + AB^2$.

90. COROLL. 1. The square of the perpendicular from the right angle is equal to the product of the two segments of the hypotenuse. The first of the three above proportions is $\therefore DP : AP : BP$. Therefore $AP^2 = DP \times BP$.

2. The square of each small side of a right-angled triangle is equal to the difference of the squares of the hypotenuse and of the other small side.

3. The difference of the squares of the two small sides is the same as the difference of the squares of the two segments of the hypotenuse.

DEM. The two small sides AD and AB being the hypotenuses of the two right-angled triangles APD, APB, their squares $AD^2 = DP^2 + AP^2$, and $AB^2 = BP^2 + AP^2$, whence $AD^2 - AB^2 = DP^2 + AP^2 - BP^2 - AP^2 = DP^2 - BP^2$.

4. In a right-angled isosceles triangle the square of the hypotenuse is double of the square of either of the two small sides. When $AD = AB$, then $BD^2 = AD^2 + AB^2 = 2AD^2 = 2AB^2$. So geometry resolves the problem of a square double of another square, the solution of which is impossible in arithmetic or algebra. A square, for instance, is but the half of the square of its diagonal. But it is to be remarked, that the diagonal and the side of the square are respectively incommensurable; that is, the side m of a square being given, and the double of its square $2m^2$, the diagonal $d =$

$d = \sqrt{2m^2}$ is incalculable, as $2m^2$ has no commensurable square root. Likewise the side $m = \sqrt{\frac{1}{2}d^2}$ is not commensurable from the given diagonal.

Geometry gives also a square quintuple or decuple of another square: The two small sides of a right-angled triangle being 1 and 2, the square of the hypotenuse is $1^2 + 2^2 = 5$ quintuple of the square of the side 1; and if the two sides were 1 and 7, the square of the hypotenuse is $1^2 + 7^2 = 50$ decuple of the other. However, quintuple and decuple squares are impossible in arithmetic and algebra, as $\sqrt{5m^2}$ or $\sqrt{10m^2}$ are incalculable roots.

91. PROBLEM 1. To trisect, quadrisection, in general to divide a given line into any given number of equal parts?

SOL. Let the line BA (Fig. 23.) be the line to be trisected. Get any line GE divided into a number of equal parts. Place the extremity of the given line BA at any of the divisions, making any angle whatsoever with EG. Take an equal number of the equal parts on both sides of A, so that $AE = AF$. Join FB, draw AP parallel to FB, the line joining FP will trisect the given line BA at O.

DEM. The two triangles AEP, FEB are similar, and give $EA : EF :: AP : FB :: 1 : 2$. The two triangles AOP, FOB are also similar, on account of two vertical angles at O, and the parallel bases AP, FB, and give $AP : FB :: AO : OB :: 1 : 2$. Bisect now OB (20.); and AB is trisected.

If you take $AG = 2AE$, the line GB parallel to AD will give $AD : GB :: 1 : 3$; and the triangles ARD, GRB will give $AR : RB :: 1 : 3$. Trisect now RB, and AB is quadrisectioned; and so on for the division of AB into any number of equal parts. The same may be done by the method of proportional lines (73.).

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92. PROBLEM

92. PROBLEM 2. To measure the distance of an inaccessible place ?

SOL. Suppose the distance to be measured from A or B to M (Fig. 24.). Take on the straight line AM a part AB as long as convenient. From A and B draw two parallel lines BG and AD terminated at DG parallel to AB. Then looking from D to the inaccessible place M, mark on BG the point F of the intersection of the visual ray DM with BG. Measure GF, DG and DA, and the two similar triangles DGF, MAD give the proportion $GF : DG :: AD : AM = \frac{DG \times AD}{GF}$

93. PROBLEM 3. To measure the vertical altitude of an object either accessible or inaccessible ?

SOL. 1. To measure the vertical height of AB (Fig. 25.) when B is accessible ?

Erect a staff OL at some distance from B, high enough to view conveniently A and B from its top O. At a small distance from OL, suspend a plummet SP, which takes spontaneously a vertical situation. Then, looking from O at A and B, mark exactly on the string of the plummet the two intersections of the visual rays F and D. The triangle OFD is similar to OAB. Measure accurately DF, OD, OB, and calculate the proportion $OD : DF :: OB : AB = \frac{DF \times OB}{OD}$

2. When B is not accessible, such as is the vertical point under the top of a mountain ?

Measure OA by the method of the second problem. Then take the horizontal visual ray from O to B. Measure accurately OF and FD, and calculate the proportion $OF : FD :: OA : AB = \frac{FD \times OA}{OF}$

Properties

Properties of Polygons, from Proportional Lines.

94. DEFINITIONS. Two polygons of the same number of sides are similar, which have their homologous angles equal, and their homologous sides proportional. Diagonals are homologous which join homologous angles. Homologous points on the sides are those which are at proportional distances from homologous angles; and the lines joining such points are *homologous lines*.

95. THEOREM 1. Similar polygons have their homologous diagonals and homologous lines proportional to the homologous sides.

DEM. Suppose the two polygons (Fig. 16.) to be similar. Draw the homologous diagonals AD and ad , AE and ae , also the homologous lines AH and ah , HR and hr . The two triangles ABD and abd are similar, having, by supposition, the angles $B=b$, contained by proportional sides. Therefore, $AB : ab :: AD : ad$, &c. Likewise, the two triangles ADH , adh , are similar, having two equal angles at D and d contained by proportional sides; whence $AD : ad :: AH : ah :: DH : dh :: BD : bd$. Again the two triangles AHR and ahr are similar and $AH : ah :: HR : hr :: BR : br :: AB : ab$, &c.

96. THEOREM 2. The oblique and the right radii of similar regular polygons are proportional to the sides and to themselves.

DEM. The two similar regular polygons (Fig. 17.) have the triangles ACB and acb formed by the oblique radii

radii similar, as well as the triangles APC and *apc* formed by the right radii. Therefore, $AB : ab :: AC : ac : CP : cp$.

97. THEOREM 3. The perimeters of similar polygons, either regular or irregular, are proportional to their homologous sides, to their oblique or right radii, to their homologous diagonals, to any of their homologous lines.

DEM. Any two similar polygons, either regular or irregular, have their homologous sides, diagonals, oblique and right radii, and any homologous lines proportional. Therefore, in the two similar polygons (Fig. 17.) we have $AB : ab :: BD : bd :: DE : de' :: EF : e'f : FG : fg :: GA : ga$. Whence $AB + BD + DE + EF + FG + GA : ab + bd + de' + e'f + fg + ga :: AB : ab :: CA : ca :: CP : cp$, &c. The two terms of the first ratio are the sums of the sides of the two similar polygons, or their perimeters.

The same may be done with the perimeters of the irregular polygons (Fig. 16.) and their homologous sides, diagonals, and any lines whatsoever.

98. COROL. The circumferences of unequal circles are proportional to their radii, to their diameters, to their homologous chords, to their homologous arcs.

DEM. A circle is a regular polygon of an infinite number of sides, having the oblique radius equal to the right one (72. 3.). Unequal circles are then similar regular polygons; and the circumferences are their perimeters. Of course, their circumferences are proportional to all their homologous lines (97.). But their diameters, their radii, their chords and arcs of the same number of degrees are their homologous lines, which therefore are proportional to their circumferences.

ferences. So a circle of a double, triple, &c. radius, diameter, homologous chord, has a double, triple, &c. circumference.

Properties of Circles, from Proportional Lines.

99. THEOREM 1. The segments of two chords intersecting each other in a circle are reciprocally proportional.

DEM. Suppose the two chords AB and FD (Fig. 26.) intersecting each other at C. Join their extremities by the lines FB and AD. The two triangles FCB and ACD are similar, on account of the two equal vertical angles at C, and the equal inscribed angles $B = D$ and $F = A$. Therefore, $FC : AC :: BC : DC$.

100. THEOREM 2. A line from any point of the circumference perpendicular to a diameter is middle proportional between the segments of the diameter.

DEM. Let AP (Fig. 26.) be perpendicular on the diameter BR. Draw from A two chords AB and AR to the extremities of the diameter. Then BAR is a right-angled triangle, BR its hypotenuse (31. 4.) and AP a perpendicular from the right angle. Whence $RP : AP : BP$ (90.).

Whence, the perpendicular $AP = \sqrt{RP \times BP}$.

101. THEOREM 3. Two secants from the same point out of the circle are reciprocally proportional to their exterior parts.

DEM. From the extremities A and B (Fig. 27.) of the two secants AS and BS, meeting at S, draw the two chords AF and BE to their intersections with the

the circumference: they make two similar triangles AFS , BES , on account of the common angle S and the two equal inscribed angles at A and B . Therefore, $AS : BS :: FS : ES$; or $ES : BS :: FS : AS$.

102. COROL. A tangent which meets a secant is middle proportional between the entire secant and its exterior part.

DEM. If the secant BS was moving about the point S , to meet the situation of the tangent TS , the points B and F approach continually, and coincide at T . Then $BS = FS = TS$, and the above proportion becomes $\therefore AS : TS : ES$or draw the two chords AT and ET : the two triangles AST and EST are similar, as they have a common angle S , besides $SAT = ETS$ (31.). Whence $\therefore AS : TS : ES$.

103. THEOREM 4. A perpendicular from any angle of a triangle to the opposite side, divides it into two such segments, as to have the opposite side to the sum of the two other sides, as the difference of the same two sides is to the difference or to the sum of the two segments (according as the perpendicular falls inside or outside the triangle).

DEM. From the angle B (Figs. 28 and 29.) of the triangle ABD , draw BP perpendicular on the opposite side AD (produced when the angle D is obtuse). From the same angle B , as a centre, describe a circle with the shortest of the two other sides BD as a radius. Produce AB to F , and AD to L . Then $DP = PL$ (14. 6.) and $AF = AB + BD$, and $AE = AB - BD$, and $AL = AP + PD$. Now the two secants AF and AD or AL give the proportion $AD : AF :: AE : AL$ (101.), which is the same as mentioned in the theorem, viz. $AD : AB + BD :: AB - BD : AP + PD$; the difference (Fig. 28.) the angle being acute, and the perpendicular inside; the sum (Fig. 29.) the angle D being obtuse, and the perpendicular

perpendicular outside. The side is in one the sum, in the other the difference of the two segments.

104. PROBLEM 1. To divide a given line into middle and extreme ratio? (86.)

SOL. Let it be required to divide the line AB (Fig. 30.) into middle and extreme ratio. On its extremity B erect a perpendicular $BC = \frac{1}{2}AB$; and with it as a radius, from the centre C , describe a circle BDE . From the other extremity A draw the secant AE through the centre C ; and from A as a centre describe an arc of a circle with the radius AD , the exterior part of the secant cutting $AF = AD$. The given line AB is divided at the point F into middle and extreme ratio.

DEM. The given line AB is a tangent to the circle DBE (18.). Therefore, $\therefore AD : AB : AE$ (102.). Then $AB - AD = AB - AF = BF : AD = AF :: AE - AB = AE - DE = AD = AF : AB$, the above proportion $\therefore BF : AF : AB$.

105. PROBLEM 2. To inscribe in a given circle a regular decagon and a regular pentagon.

SOL. Divide the radius of the given circle into middle and extreme ratio; and a chord equal to its greater segment is the side of the regular inscribed decagon.

DEM. Take the line AB (Fig. 30.) which, in the last, has been divided into middle and extreme ratio at the point F ; and from the centre A , with the radius AB , describe a circle LBG . On the small segment BF construct an isosceles triangle BLF , with $BL = FL = AF$ the great segment, join the vertex L and A . Then AFL is an isosceles triangle, having $AF = FL$; and the two triangles BLF and BAL are isosceles and similar, on account of their common angle B contained by proportional sides $BF : BL :: BL : BA$; whence $AL = AB$;

AB; and **L** is on the circumference of the circle **LBG**. Now the chord **BL** is the side of the regular inscribed decagon. For the angle $\text{BAL} = \text{BLF} = \text{ALF}$ is the half of the angle **ABL**, and also the half of the angle **ALB**. Therefore, the angle **BAL** is the fifth part of the sum of the three angles of the triangle **BAL**; that is, $\frac{180^\circ}{5} = \frac{360^\circ}{10}$; and the arc **BL**, which measures that central angle **BAL**, is $\frac{1}{10}$ part of the circumference; and its chord, the side of the regular inscribed decagon. The chord of a double arc will be the side of a regular inscribed pentagon.

106. **PROBLEM 3.** To graduate a circle geometrically?

SOL. From any point of the circumference draw the side of the regular inscribed hexagon, the chord of an arc of 60° . From the same point draw the side of the regular inscribed decagon, a chord of 36° . Their difference is an arc of 24° , which, bisected three times, gives an arc of 3° . For want of the geometrical trisection of a given angle, a problem unresolved to this day, geometry cannot bring farther than 3° the graduation of a circle, so very important for the construction of geometrical and astronomical instruments. Beginners, however, will be gratified to find here subjoined a solution of that celebrated problem, which, though not a geometrical one, may be of use in practical geometry.

107. **PROBLEM 4.** To trisect a given acute angle?

SOL. Let it be required to trisect the angle **BAD** (Fig. 31.). From any point **C** of either of the two sides **AD** describe a circle by the vertex **A**. From the centre draw a radius parallel to the other side **AB** and produce it towards **S**. A secant from the vertex **A** meeting

meeting the prolongation of that radius, and having its exterior part LS equal to the radius CL will trisect the given angle BAD .

DEM. The triangle CLS is isosceles by supposition; the triangle ACL is also isosceles, having $CA = CL$. Therefore the angle CAL is equal to the angle ALC , which being exterior to the isosceles triangle CLS is double of the angle $ASC = BAS$ (49.) then the angle BAS is the half of the angle SAD . Bisect this last; and the given acute angle BAD is trisected.

This method of trisecting the angle is not geometrical: as the secant is not known geometrically, the exterior part of which is equal to the radius of the circle. Were the point L determined, at which the secant is to intersect the circumference, to have its exterior part LS equal to the radius CL ; or at which the chord FL parallel to the side AB should bisect the angle ALC made by the radius and the interior part AL of the secant; the problem should be resolved geometrically.

But this affords a mechanical method of trisection in practical geometry, by means of the instrument $EACLS$. It consists of three rules making double compasses, viz. Two equal rules AC and CL moveable about the point C , and the rule ES connected with the rule CL , and moveable about the point L , so that $LS = CL = CA$. The rule CA is applied on the side AD , so that its extremity A is exactly at the vertex. Then the rule ES is made to slide contiguously to the extremity A until its extremity S comes to the parallel CS . The edge of the rule ES is the secant required.

108. THEOREM 5. The radius of the circle being expressed by 1; if 2 be added to the *supplemental* chord of an arc, the square root of the sum is equal to the supplemental chord of half the same arc.

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The supplemental chord of an arc is the chord of its supplement.

DEM. Take the arc AB (Fig. 32.) and bisect it at F, draw the diameter AD, the radius CF, and the chords AF, FB, AB, FD, BD. The chord BD is the supplemental chord of the arc AB; and FD the supplemental chord of the arc $AF = \frac{1}{2}AB$. From F to the prolongation of the diameter AD, draw $FL = FD$. Then the isosceles triangles DFL, DCF are similar, on account of their common angle $FDL = DLF = DFC$; whence $\therefore DC : DF : DL$. But the two triangles $LAF = DBF$; because the angles $BDF = ADF = ALF$, the sides $AF = BF$, and the angles $DBF = LAF$ being both supplement to the same angle DAF, as AFBD is an inscribed quadrilatre (64.2.). Therefore, their homologous sides $BD = AL$. Now, as $DL = DA + AL$, and $DA = 2DC$, the above proportion $\therefore DC : DF : DL$ becomes $DC : DF :: DF : 2DC + DB$. Whence $DF^2 = 2 \times DC^2 + DC \times DB$: and as the radius $DC = 1$, $DF^2 = 2 \times 1^2 + 1 \times DB = 2 + DB$; and $DF = \sqrt{2 + DB}$; which is, the supplemental chord DF of half the arc AB is equal to the square root of the supplemental chord DB of the entire arc AB increased by 2.

109. COROL. Hence, the supplemental chord of any arc being once calculated, may be taken as a standard to calculate from it the supplemental chords of that arc, bisected and subbisected until it is infinitely small and coinciding sensibly with its chord.

The supplemental chord of an arc of 60° is easily calculated for such a standard, it is equal to $\sqrt{3}$.

DEM. Suppose the arc $AB = 60^\circ$, its chord $AB = AC$ (47.). Draw its supplemental chord BD; and ABD is a right angled triangle (31. 4.), in which $BD^2 = AD^2 - AB^2$ (90. 2.); and as $AC = 1$, and $AD = 2$, $BD^2 = 2^2 - 1^2 = 4 - 1 = 3$; and $BD = \sqrt{3} = 1.7320508076$ to 10 decimal figures.

Now

Now adding 2 to this last, and extracting the square root of the sum $\sqrt{3.7320508076} = 1.9318516525$ is the supplemental chord of an arc of 30° sustained by the side of a regular polygon of 12 sides. The square root of this last increased by 2 will be the supplemental chord of an arc sustained by the side of a regular polygon of 24 sides. The sixth similar extraction of the square root, after this last, will give for the supplemental chord of the arc sustained by the side of a regular polygon of 1536 sides $\sqrt{3.9999832669}$, which is an arc of $0^\circ 14' 3'' 45''' < \frac{1}{4}1^\circ$. Such a regular polygon inscribed in a circle, having something more than 4 of its sides under an arc of 1° , may be taken as coinciding sensibly with the circumference; and the ratio of its right radius to its perimeter, the same as the ratio of the radius to the circumference of its circumscribed circle. Hence,

110. PROBLEM 5. To calculate an approximate ratio of the radius or diameter to the circumference?

SOL. Suppose AF to be the aforesaid arc of $0^\circ 14' 3'' 45'''$ sustained by the side AF of a regular inscribed polygon of 1536 sides, and DF its supplemental chord. The right-angled triangle AFD (31. 4.) gives $AF^2 = AD^2 - DF^2$ (90. 2.), and, as $AD = 2$, $AF^2 = 4 - 3.9999832669 = 0.0000167331$; and $AF = \sqrt{0.0000167331} = 0.0040906112$ the length of the side of the regular inscribed polygon of 1536 sides, the $\frac{1}{1536}$ part of its perimeter and of the circumference of the circumscribed circle. Therefore, $0.0040906112 \times 1536 = 6.2831788$ is the circumference of the circle, its radius being 1. Such is then the ratio of the radius to the circumference, 1 to 6.2831788, and of the diameter to the circumference 1 to 3.1415894.

This

This ratio must obviously be somewhat too small, as the arc is greater than the side of the polygon which sustains it, whatever be the number of its sides. From the earliest age of geometry, the most able geometers have tried the solution of the problem by various methods of elementary and of transcendent mathematics. Archimedes stated the ratio of the diameter to the circumference to be 7 to 22, or rather to a number between 21 and 22. Adrian Metius gave it 113 to 355. The most accurate of all, that of Ludolph de Ceulen, calculated to 36 decimal places, is 1 to 3, 141592653, &c. which differs from the above only at the fifth decimal place. The approximation is so near the true ratio, that in a circle of a radius of 1000000 of feet, the error on the circumference should scarce be 1 foot. From this last, Euler calculated the radius of the circle to be equal to an arc of the same circle of $57^{\circ} 17' 44'' 48''' 22''''$ &c. which gives for an arc of 1" the $\frac{1}{206265}$ part of the radius.

111. As the circumference contains 360° , the length of an arc of 1° is found by the proportion $360^{\circ} : 6,2831788 :: 1^{\circ} : 0,01745$. Whence the length of an arc of any given number of degrees may be found by multiplying the given number of degrees by 0,01745. So the length of an arc of 60° is $0,01745 \times 60 = 1,047$ greater than the radius, which is its chord, by 0,047. The arc of $1'$ is $0,01745 \times \frac{1}{60} = 0,000290$. And the arc of $1''$ is $0,01745 \times \frac{1}{3600} = 0,000005 = \frac{1}{200000}$ nearly,

SECTION

SECTION II.

OF SURFACES.

112. A SURFACE or area is, as was said before (1.) a part of extension consisting of only two dimensions, length and breadth, abstracting from thickness; and may be considered as formed by the motion of a line; or as a series of contiguous lines of an infinitely small breadth, its immediate elements. If the generating line be a straight one, and moves in a straight direction, the surface formed is a *plane*; and therefore, a straight line, such as a rule, placed on a plane, touches it by all its points in every direction. But a straight line moving in a curve direction, or a curve line moving in a straight or curve direction, describes a curve surface.

The object of this section is : 1. The measurement of plane surfaces. 2. The relation of their magnitudes. And, 3. The planes. The curve surfaces belong to the section of solids.

Measurement

Measurement of Surfaces.

113. **MAGNITUDES** of any kind whatsoever are generally measured by determining how often they contain a magnitude of the same kind agreed upon as a unit and as a standard of measure. The unit of surface generally taken for standard in practical geometry is a square surface of a determinate magnitude, such as one square inch, foot, yard, perch, mile, &c. according to circumstances. In abstract geometry a surface may be measured by the sum of the elements of surface; that is, by the sum of the straight lines which it contains, considering them as having an infinitely small breadth.

114. **THEOREM 1.** The surface of a rectangle is measured by the product of two of its contiguous sides, one the *base*, the other the *altitude* of the rectangle.

DEM. Let BEFD (Fig. 33.) be a rectangle or right-angled parallelogram, and the small square A the unit or standard of measure. It is obvious that, if the base EF and the altitude BE be divided into parts equal to the common side of the small square A, and lines be drawn parallel to the base and altitude from the divisions of each to the opposite side, these lines will form within the limits of the rectangle a number of small squares each equal to the standard A, which is the product of the divisions of EF multiplied by the divisions of BF. Therefore, &c.

Again, the area of the rectangle BEFD is equal to the sum of the elementary lines filling the surface. But every elementary line is equal to the base EF, as they

they are all parallels between the two parallel sides BE and DF ; and the number of elementary lines is equal to the number of elementary points in the altitude BE . Therefore their sum is equal to one of them multiplied by their number, that is, to the base EF multiplied by the altitude BE .

Hence the product of two quantities is called their rectangle.

115. COROLL. 1. The surface of an oblique-angled parallelogram is measured by the product of one of its sides taken for its base, multiplied by the perpendicular from the side opposite to the base, which is its altitude.

DEM. The surface of the oblique-angled parallelogram $LEFG$ (Fig. 33.) is equal to the rectangle $BEFD$, on account of the triangles $BEL = DFG$ (58.), one subtracted from, the other added to the rectangle. Then, its surface is as above $EF \times BE \dots$. Besides, the rectangle and the parallelogram have an equal number of equal elementary lines, all equal to EF , and as many as there are points in the perpendicular BE . Then, its surface is $EF \times BE$.

2. Parallelograms are of equal surfaces which have equal bases and equal altitudes; which, having the same or equal bases, are between parallels; which have their bases reciprocally proportional to their altitudes.

3. A square being a rectangle of four equal perpendicular sides, its surface is equal to the square of one of its sides. Hence, the name of square to the second power of any quantity.

4. The geometrical square of a binom consists of the same parts as its numerical square.

DEM. Let the side AB (Fig. 34.) of the square BD consist of two parts $AE + EB$. Take $DF = AE$,
and

and draw EG and FL parallel to AD and DC. Then FG is the square of AE; BO the square of EB; and the two equal rectangles AO and CO, the double product of AE, multiplied by EB.

The same is true of the square of any geometrical polynom. Let AB be divided into any number of parts, for instance, into three; $AE=a$, $EP=b$, $PB=d$. Then, the square $BD=AB^2=(a+b+d)^2=a^2+2ab+2ad+b^2+2bd+d^2$. For it is obvious that $FG=a^2$; $RF+SG=2ab$; $AR+CS=2ad$, $RS=b^2$; $PR+KS=2bd$; $PK=d^2$.

116. THEOREM 2. The measure of the surface of a triangle is half the product of any of its sides, taken for its base, multiplied by its perpendicular from the vertex, the altitude of the triangle.

DEM. Any triangle, such as ABD (Fig. 35.), may be completed into a parallelogram, by a line BF from the vertex B, parallel and equal to the base AD, and the line DF parallel and equal to AB. Then, BP perpendicular to AD is the altitude of the triangle ABD, as well as of the parallelogram AF. The triangle is half the parallelogram (61.), therefore the surface of

$$ABD = \frac{1}{2}AF = \frac{AD \times BP}{2} \quad (62.) = \frac{1}{2}AD \times BP = \frac{1}{2}BP \times AD.$$

Again, the elements of surface from the base AD to the vertex B are as many as there are points in the altitude BP and decrease in an arithmetical progression, as is easily demonstrated by (Fig. 18.). Therefore, their sum, which is the surface of the triangle ABD, is the product of half the sum of the extreme elements $AD+B$ multiplied by $BP = \frac{AD \times BP}{2}$, as the element B of the vertex is infinitely small.

If

If EG be drawn parallel to the base AD , and bisecting the altitude BP , it may be taken for half the base AD ; since the similar triangles EBG , ABD give $BR : BP :: EG : AD :: 1 : 2$ (75.) and the surface of ABD is $EG \times BP$.

117. COROLL. 1. Triangles have their surfaces equal, when they have equal bases and equal altitudes; when they have the same or an equal base, and are contained by parallels; when they have their bases and altitudes reciprocally proportional. The same as for parallelograms (115. 2.).

2. The surface of a right-angled triangle is equal to half the product of the two perpendicular sides, containing the right angle; one being its base; the other, its altitude.

3. A triangle and a parallelogram of the same or an equal base, with the same or an equal altitude, or between the same or equidistant parallels, have their surfaces as 1 to 2.

118. THEOREM 3. In a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the two small sides. The same as (89.) geometrically.

DEM. Let BG , AE , AL (Fig. 38.) be the squares of the three sides of the triangle BAD rectangle at A . Draw AP parallel to BC , and join AC , AG , BE , DL . Then is the triangle $LBD = ABC$, on account of $AB = LB$, $BD = BC$, and the angles $LBD = ABC$, each, the angle ABD added to the right angle DBC in one, to ABL in the other (58.). Besides, the surface of the triangle LBD is half the surface of the square AL , on account of the same base BL parallel to MD (107. 3.); and, for the same reason, the triangle ABC is half the rectangle CO . Therefore, $AL = CO$. By a similar demonstration, $AE = GO$. Then, $CO +$

I

 $GO =$

$GO = BG = AL + AE$. The square of the hypotenuse equal to, &c.

119 THEOREM 4. The measure of the surface of a trapezium is half the product of the sum of its two parallel sides multiplied by their perpendicular distance or the product of the same perpendicular by the middle element between the two parallel sides.

DEM. The trapezium ABDF (Fig. 36.) has its two sides BD and AF parallel; and the diagonal AD divides it into two triangles BAD, DAF, which, being between parallel bases, have the same altitude OP perpendicular to their two bases BD and AF. Therefore the surface of the trapezium is the sum of the surfaces of the two triangles $BAD + ADF = \frac{1}{2}BD \times OP + \frac{1}{2}AF \times OP = \frac{BD + AF}{2} \times OP$.

Again, if the sides BA and DF be produced to meet at S, the trapezium is the triangle BSD truncated at AF, and, of course, its elements of surface are an arithmetical progression between the extreme terms BD and AF; and their number is the perpendicular OP. Whence, their sum, which is the surface of the trapezium, is $\frac{BD + AF}{2} \times OP$.

Then, if ML be drawn parallel to BD and bisecting OP, it is the middle term of the arithmetical progression, and, of course, equal to half the sum of the extremes. Therefore, the surface of the trapezium, is $\frac{BD + AF}{2} \times OP = ML \times OP$.

And, by analysis. Suppose SP perpendicular from the vertex S to the two parallel bases of the trapezium. Take $BD = a$, $AF = d$, $SO = s$, $OP = r$. Then $ABDF = BSD -$

$$BSD - ASF = a \times \left(\frac{r+s}{2} \right) - \frac{d \times s}{2} \text{ (106.)} = \frac{ar + as - ds}{2}$$

But $a : d :: r + s : s$ (85.), and $as = dr + ds$. Thence

$$\frac{ar + dr + ds - ds}{2} = \frac{ar + dr}{2} = \frac{a+d}{2} \times r = ABDF.$$

120. THEOREM 5. Half the product of the perimeter multiplied by the right radius, measures the surface of a regular polygon.

DEM. The oblique radii divide a regular polygon (Fig. 17) into as many equal triangles as there are sides. The bases of all the triangles are the sides themselves, and the right radius is their common altitude (69.). The surface of each triangle is half the product of its base by its altitude (106.). Therefore half the product of the sum of the sides, which is the perimeter of the polygon, multiplied by the right radius is the surface of the polygon.

121. COROL. The surface of an inscribed polygon increases when the number of its sides increases.

DEM. The sides of an inscribed polygon increasing in number, become smaller chords, remove from the centre, and their sum and the right radius increase; whence, the surface of the polygon increases. So, if the hexagon (Fig. 17) was to become a polygon of 12 sides, each triangle like BCD receives an increase of a small triangle like BRD.

122. THEOREM 6. Half the product of the circumference multiplied by the radius, measures the surface of the circle.

DEM. A circle is a regular polygon of an infinite number of sides, in which the oblique and right radius are the same (72. 3.). Of course, the surface of the circle is equal to half the product of its circumference by its radius.

123. COROLL.

123. COROLL. 1. The surface of the circle is the $\frac{1}{4}$ part of the product of its circumference by its diameter. Whence the circumference being 3,14159, when the diameter is 1 (110) the surface of a circle, the diameter of which is taken equal to 1, will be $\frac{1 \times 3,14159}{4} = 0,7854$.

2. The surface of a sector is equal to half the product of its arc, by the radius of the circle. Because, a sector of a circle is a sum of infinitely small triangles, which have the radius for their common altitude, and the arc for the sum of their bases.

3. The surface of a segment is equal to the difference between the surfaces of the sector under its arc and of the triangle under its chord.

4. The surface of a circular ring contained between the circumferences of two concentric circles, is the difference of the surfaces of the two circles.

Geometry measures the surfaces of the above figures only, namely, triangles, parallelograms, trapeziums, regular polygons and circles. Irregular figures are measured by dividing them into triangles or parallelograms, which constitutes the art of surveying. Therefore,

124. PROBLEM 1. To survey a given surface?

SOL. *Surveying* is the art of measuring the surfaces of lands, which are mostly irregular figures. There are many treatises on surveying, wherein various methods of measurement are found. They are not the object of these elements; and therefore we shall confine ourselves to indicate the most common and safest method of surveying. It consists in dividing the land to be surveyed into triangles or parallelograms, which are measured separately, and made into one sum. Such a division is arbitrary,

arbitrary, according as the local circumstances render the surveying more convenient and expeditious. Right-angled triangles or rectangles are commonly used; because their perpendicular sides, which are their bases and altitudes, are given at once by the common instrument of the surveyor.

125. PROBLEM 2. To reduce an irregular polygon to a triangle of an equal surface?

SOL. From the angle F (Fig. 37.) of the irregular polygon ABDGF draw the diagonal FD; and from the angle G, the line GE parallel to FD, and meeting the side BD produced. Join FE; and the quadrilater AB EF is equal to the pentagon ABDGF. For, the two triangles FGD = FDE, on the same base FD, and between the parallels FD, GE (107.). Their common part FCD being subtracted from each, remain DCE = FCG, the first added to, the second subtracted from the pentagon to form the quadrilater. Now, draw the diagonal AE, and its parallel FL to the same BD produced. Join AL, and the triangle BAL is equal to the quadrilater AB EF. For, the two triangles LAF = FEL. Subtracting from both, their common part FOL, remain EOL = AOF, one added to, the other subtracted from the quadrilater, to change it into the triangle ABL equal to the pentagon ABDGF.

126. PROBLEM 3. To find the quadrature of a given figure?

SOL. To square or find the quadrature of a given figure is to find the side of a square of an equal surface. The measure of any surface whatever is the product or rectangle of two of its dimensions only, as is obvious from the above theorems. Therefore, the mean geometrical proportional between the two dimensions is the side of a square of an equal surface. For instance, if a and b be the two dimensions, the
product

product of which measures the surface s , then $s=ab$. Wherefore, if I take $\div a : x : b$, $ab=x^2=s$, and $x=\sqrt{ab}$, the side of a square of the same surface as the figure of which ab measures the surface. Hence,

127. The surface of a parallelogram is equal to that of a square the side of which is the mean geometrical proportional between its base and altitude. And the surface of a triangle or of a trapezium is equal to the square of a line mean geometrical proportional between the altitude and the middle element (106. 109.).

128. The surface of a regular polygon is equal to the square of a line mean proportional between its perimeter and half the right radius, or between half the perimeter and the right radius (110.). But this requires the ratio of the right radius to the side, which may be found by the properties of the right-angled triangle. For instance, in the equilateral triangle BEG (Fig. 17.) the oblique radius CB being 1, the right radius $CL=\frac{1}{2}$ (71.) and the square of BL or $\frac{1}{2}BG=1^2-(\frac{1}{2})^2$ (90. 2.) $=1-\frac{1}{4}=\frac{3}{4}$; and $BL=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}=\frac{1,73205}{2}=0,86603$.

Whence, half the perimeter of the equilateral triangle BEG is $3 \times 0,86603$; and its surface is $3 \times 0,86603 \times \frac{1}{2}=1,29905$ (116.). Now, to square it, make $\frac{1}{2} : x :: x : 3 \times 0,86603$; and the side of a square of the same surface as the equilateral triangle is $x=\sqrt{3 \times 0,43302}=1,13976$; and the surface of the square is $(1,13976)^2=1,29905$.

In the regular hexagon (Fig. 17.) the oblique radius CB being 1, half the side AB or $BP=\frac{1}{2}$ (70.) and the right radius $CP^2=BC^2-BP^2=1^2-(\frac{1}{2})^2=\frac{3}{4}$, and $CP=\frac{\sqrt{3}}{2}=0,86603$, which multiplied by half the perimeter equal to $\frac{1}{2} \times 6=3$ gives for the surface 2,59809, double

double the surface of the inscribed equilateral triangle, as is geometrically obvious from the figure, in which the inscribed hexagon and the inscribed triangle contain, the first 6, the second 3 equal triangles, such as BAG. Now, to square it, make $0,86603 : x :: x : 3$; and calculate $x = \sqrt{3 \times 0,86603} = \sqrt{2,59809} = 1,611859$.

So, the geometrical quadrature of regular polygons is given only by approximation. In practical geometry the dimensions are taken mechanically.

129. The surface of a circle is equal to that of a square, the side of which is a mean geometrical proportional between the radius and half the circumference. It is the celebrated problem of the quadrature of the circle. The solution requires the ratio of the radius or diameter to the circumference, which is to be had only by approximation. Such a ratio has been calculated (110.) and found to be, of the radius to the circumference as 1 to 6,2831788, of the diameter to the circumference as 1 to 3,1415894. Wherefore taking $1 : x :: x : 3,14159$, the side of a square of the same surface as the circle, is $x = \sqrt{3,14159}$, and the surface of the square is 3,1416, the same as above (123.) where the surface of the circle is taken by multiplying the circumference by $\frac{x}{4}$ part of the diameter.

To find the area of a circle the radius of which is 10?

The proportion $1 : 10 :: 3,14159 : c$ gives half its circumference $c = 31,4159$, which multiplied by the radius 10, gives 314,159. The side of an equal square will be found by $10 : x :: x : 31,4159$, in which $x = \sqrt{314,159} = 17,7245$ nearly.

Relations

Relations of Surfaces.

THE foregoing measurement of surfaces is the estimate of their absolute magnitude. The relations of surfaces are their relative or proportional estimate.

130. THEOREM 1. The surface of any figure whatsoever is equal to the product of two of its dimensions.

This theorem is demonstrated by the foregoing theorems, from (114.). So, calling the surface of any figure S , and the two dimensions the product of which measures the surface, A and B , the surface $S=A \times B$; and if the surface of another figure is called s , and the two dimensions a and b , the surface $s=a \times b$.

131. COROLL. 1. The surfaces of two figures are proportional to the products of their two dimensions.

DEM. The dimensions of a surface S being A and B , and of a surface s being a and b , $S=AB$, and $s=ab$. Therefore, $S : s :: AB : ab$.

2. If two figures have $A=a$, then $S : s :: B : b$. If they have $B=b$, then $S : s : A : a$. And if they have $A : a :: b : B$, then $AB=ab$, and $S=s$. For instance, the surfaces of a triangle and a trapezium of the same altitude are proportional to their middle elements. And if their middle elements are reciprocally proportional to their altitudes, their surfaces are equal.

132. THEOREM 2. The surfaces of any two similar figures are proportional to the squares of any two of their homologous dimensions.

DEM.

DEM. All homologous dimensions or lines of similar figures are proportional (95.). Calling then the dimensions the products of which measure the surfaces S and s of two similar figures, A, B and a, b ; and any other two homologous dimensions D and d ; we have $A : a :: B : b :: D : d$. Therefore, the ratio of their surfaces is the compound ratio of two equal ratios, which is the duplicate ratio of the simple ratios; viz. $S : s :: AB : ab :: A^2 : a^2 :: B^2 : b^2 :: D^2 : d^2$.

133. COROLL. 1. The surfaces of similar parallelograms, or of similar triangles are in the duplicate ratio of their bases and altitudes. The surfaces of similar polygons are in the duplicate ratio of their homologous sides, oblique or right radii, perimeters, diagonals, &c. The surfaces of circles (which are similar regular polygons) are in the duplicate ratio of their radii, diameters, circumferences, chords of arcs of the same number of degrees, &c.

2. As the surface of a circle s , the diameter of which being 1, is 0,7854 (123.) the surface S of any other circle, the diameter of which is D , will be found by multiplying the square of that diameter D , by 0,7854. Because $d^2 : D^2 :: s : S = \frac{D^2 \times s}{d^2} = \frac{D^2 \times 0,7854}{1^2} = D^2 \times 0,7854$. For instance, the surface of a circle of a diameter $D=20$ is $400 \times 0,7854 = 314,159$, as above (129.).

3. The surfaces of inscribed and circumscribed equilateral triangles are as 1 to 4. Of inscribed and circumscribed squares as 1 to 2. Of an inscribed and circumscribed hexagon as 3 to 4.

DEM. As the side of a circumscribed regular polygon is a tangent (66.), its right radius is the radius of the circle, equal to the oblique radius of the inscribed similar polygon. Therefore, for the inscribed

K

and

and circumscribed equilateral triangles, the right radii are as 1 to 2 (71.); and their surfaces as 1^2 to 2^2 , as 1 to 4. For the inscribed and circumscribed squares; the side of the circumscribed square is equal to the diagonal of the inscribed one (the diameter of the circle, the hypotenuse of an isosceles right-angled triangle on the two sides of the inscribed) the square of which is to the square of the side as 2 to 1 (90. 4.). For the inscribed and circumscribed regular hexagons; the right radius of the inscribed one is $\sqrt{\frac{3}{4}}$ and the right radius of the circumscribed one is 1 (128.). Therefore, their squares as $\frac{3}{4}$ to 1, as $\frac{3}{4}$ to $\frac{4}{4}$, as 3 to 4.

134. THEOREM 3. The three sides of a right-angled triangle being homologous sides of three similar figures (87.) the surface of any figure whatsoever on the hypotenuse is equal to the sum of the two surfaces of the similar figures on the two small sides.

DEM. Let the hypotenuse be expressed by A , and the surface of its figure by S ; the two small sides by a and b , and the surfaces of their similar figures by s and f . Then, as the three sides of the right-angled triangle are homologous sides of similar figures, $S : A^2 :: s : a^2$; and $s : f :: a^2 : b^2$; whence, $s + f : a^2 + b^2 :: s : a^2$; and $S : A^2 :: s + f : a^2 + b^2$, or $S : s + f :: A^2 : a^2 + b^2$. But $A^2 = a^2 + b^2$ (89.). Therefore, $S = s + f$.

135. COROLL. 1. In a right-angled triangle ABD (Fig. 39.) if a perpendicular BP be drawn from the right angle B to the hypotenuse AD; the square or any other figure on the hypotenuse, is to the square or any other similar figure on either of the small sides, as the hypotenuse is to the segment contiguous to the small side.

DEM.

DEM. The three triangles ABD, APB, BPD are similar figures (87.). Then, the surfaces $ABD : APB :: AD^2 : AB^2$; and the surfaces $ABD : BPD :: AD^2 : BD^2$. But the three triangles having a common altitude BP on their three bases AD, AP, PD, their surfaces are as their bases (131. 2.). Therefore, $AD^2 : AB^2 :: AD : AP$; and $AD^2 : BD^2 :: AD : DP$. And if three similar figures be constructed on the three sides, the surfaces of which are, S on AD, s on AB, f on BD; the proportions become $S : s :: AD : AP$, and $S : f :: AD : DP$.

2. The squares of two chords AB and AC (Fig. 39.) from the same extremity A of a diameter AD are proportional to the corresponding segments of that diameter.

DEM. As $AD^2 : AB^2 :: AD : AP$, so $AD^2 : AC^2 :: AD : AF$. Whence, $AD^2 \times AP = AB^2 \times AD$; and $AD^2 \times AF = AC^2 \times AD$; from which $AB^2 : AC^2 :: AP : AF$.

OF PLANES.

136. DEFINITION. A *plane* is a surface with which all the points of a straight line, such as a rule, are in contact in every direction. Hence,

137. COROLL. 1. A straight line is entirely in a plane in which it has two of its points.

2. Three points, which are not in the same straight line, are requisite and sufficient to fix the position of a plane. Whence, two planes coincide, if they have three common points which are not in the same straight line: and the three sides of a triangle are in the same plane.

3. A plane may be considered as formed by the motion of a straight line, which runs contiguously to two concurring or parallel lines. A plane may also be considered as formed by a line which revolves about another immoveable perpendicular line.

4. Within the limits of a figure, the plane surface is the smallest, and of an invariable extension. Because the sum of its elementary lines is the smallest, and constant. Whence, a plane surface is the only proper measure of extension within the limits of a figure.

What is to be considered here is, the relative position of lines with planes, and of planes with planes.

Relative

Relative Position of Lines with Planes.

A LINE may be perpendicular, oblique, or parallel to a plane, and make with it angles of different magnitudes.

138. A line is *perpendicular* to a plane, if it diverges equally from it in every direction; or, which is the same, if it makes right angles with all the lines in the plane that pass through its meeting point.

139. THEOREM 1. A line is perpendicular to a plane, if it makes right angles with any two different lines in the plane, drawn from the meeting point.

DEM. Suppose the line AC (Fig. 40.) erected on the point C of the plane MSGR, and to make right angles with CD and CK, two lines in the plane drawn from the meeting point. The line AC is then also perpendicular on BD and KL, the prolongations of CD and CK. Now, if the line AC was perpendicular to every line in the plane passing through its meeting point C, it should be perpendicular to BD and KL. But it cannot keep the same meeting point C and lose its perpendicularity to any, without changing its situation, and losing its perpendicularity with either BD or KL. Therefore, if it be perpendicular to CD and CK, it is perpendicular to all the lines in the plane that pass through the meeting point C, and, of course, to the plane.

140. COROLL. 1. If a circle be described on the plane from the meeting-point C, every point of the perpendicular AC may be taken as the centre of that circle.

circle. Because, all the oblique lines from any point of AC to the extremities of all the diameters of the circle, such as B, D, K, L, are equal (14. 4.).

2. Two lines in the plane bisecting perpendicularly two chords of a circle such as BKDL, described on the plane from the extremity A, or from any other point of the line AC as a centre intersect each other at the meeting point C of that perpendicular (24.). Thence,

3. A method of drawing a perpendicular to a plane from any given point, such as A out of that plane, viz. From the given point A, as a centre, describe a circle on the plane, and bisect two of its chords by perpendiculars: the line joining the intersection of these two perpendiculars and the point A is the perpendicular from A to the plane: and that perpendicular is the true measure of distance from the exterior point A to the plane.

141. A line is *oblique* to a plane, if it diverges unequally from it in different directions; or if it makes unequal angles with any of the lines in the plane that pass through its meeting point. The line AC, for instance, being erected on the plane at the point C, so as to diverge more from the side KBL than from the side KDL, will obviously make unequal angles with every line in the plane drawn from the meeting point C. To have its true obliquity to the plane, an angle must be found of an invariable magnitude; which is determined as follows:

142. THEOREM 2. The true measure of an angle made by an oblique line with a plane is the same as that of the angle it makes with a line in the plane, joining its meeting, and the meeting of a perpendicular, from its extremity or any other of its points to the plane.

from

DEM. Suppose the line AC erected on the point C, of the plane GSMR, so that AP is a line from the point A perpendicular to the plane. The angles it makes with the various lines CD, CO, CL, &c. are measured by arcs of circles described from the meeting point C, sustained by chords from A to D, to O, to L, &c. Such chords are the hypotenuses of right-angled triangles APD, APO, &c. in all which AP is a constant side, and the sides PD, PO, &c. are variable sides. The side PD is the shortest of all; since $CP + PD = CO$ is shorter than $CP + PO$. Therefore the hypotenuse or chord AD is the shortest of all; and the arc it sustains, which is the measure of the angle ACD, is the smallest. Of course, the angle ACD made by the oblique line AC with the line CD joining its meeting and the meeting of its perpendicular AP, is the smallest of all the angles; it is of an invariable magnitude, and therefore the true measure of its obliquity with the plane.

The angle ACB made by the oblique line AC with the prolongation of CD is the supplement of the angle ACD, and, of course, of an invariable magnitude. But of the two, the acute one is taken for the measure of the obliquity of the line with the plane.

143. COROL. A straight line which passes through a plane makes equal vertical angles with the two opposite sides of the plane. Because, the plane having no thickness, the angles made by the oblique line AC with the two opposite sides of the plane are the same as the angles it makes with CD and its prolongation CB (9. 3.).

144. A line is *parallel* to a plane with which it has no divergency. Of course, the line and the plane can never meet, though infinitely produced.

145. THE-

145. **THEOREM 3.** A line is parallel to a plane, if two lines from it perpendicular to the plane are equal.

DEM. Such a line is parallel to a line in the plane, which passes by the meetings of the two perpendiculars (16. 2.). Therefore, it can never meet it: and as that *corresponding* line in the plane is the nearest to it, since it passes through the meetings of the perpendiculars (15.) it can never meet the plane, and is parallel to it.

Relative Position of Planes.

THE relative position of planes, like the relative position of lines, consists in being parallel to one another, so as never to meet, though infinitely produced; or in concurring perpendicularly or obliquely.

146. **THEOREM 1.** Two planes are parallel if they have between them three equal perpendicular lines, not in the same straight line. Or, which is the same, if they have three corresponding points equidistant, not in the same straight line.

DEM. Suppose the two planes MN and *mn* (Fig. 41.) erected on the plane of the plate, and the three lines Aa, Bb, Dd, equal, and also perpendicular to both planes. Then Aa is perpendicular to AB and AD, to *ab* and *ad*; Bb is perpendicular to BA and BD, to *ba* and *bd*; and Dd is perpendicular to DA and DB, to *da* and *db* (139.). Therefore, the three sides of the triangle ABD are equal and parallel to the three homologous sides of the triangle *abd* (16. 2.); and the sides of one can never meet the sides of the other. Hence,
the

the two planes MN and mn , in which the two triangles are, can never meet, and are parallel.

147. COROLL. 1. The same may be said of planes which have between them three equal parallel oblique lines, or even three equal equally oblique lines, not in the same straight line, viz. the planes are parallel; because, equal oblique lines of the same obliquity suppose equal perpendicular lines between the planes (14. 4.).

2. A line makes, with two parallel planes, equal corresponding, equal alternate interior, and equal alternate exterior angles.

3. Two parallel lines are equal, and are either both perpendicular, or both of the same obliquity between two parallel planes; and reciprocally.

148. THEOREM 2. The intersection of two planes is a straight line.

DEM. When two planes intersect each other, a straight line has two of its points common to the two planes; therefore, it has all its points in each of the two planes (137.); and is itself their intersection.

149. THEOREM 3. The two lines of intersection of the same plane with two parallel planes are parallel.

DEM. Suppose the two planes MN and mn (Fig. 41.) to be parallel, and a third plane $AabB$ to intersect them at AB and ab . From any two points of the intersection AB draw in the third plane $AabB$ two parallel lines; for instance, Aa and Bb . These two lines are equal (147. 3.). Whence, the two lines of intersection AB and ab join corresponding points of parallel lines, and are equal (16. 2.).

150. COROL. Three or more parallel lines between two parallel planes describe equal triangles, or equal polygons on the two planes.

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DEM.

DEM. By the last, the intersections AB and ab of the plane Ab are equal, in the supposition of Aa and Bb being parallel. Then, likewise, $AD = ad$, and $BD = bd$, in the supposition of the three lines Aa , Bb , Dd , being parallel. Therefore, the two triangles $ABD = abd$, on account of their three sides being equal (56.).

When there are more than three parallel lines, the equality of the polygons is demonstrated like the equality of the triangles, by drawing corresponding diagonals, which divide the two polygons into triangles.

151. THEOREM 4. The measure of the angle made by two planes, which meet or intersect each other, is the same as of the angle made by two lines in the two planes, which are both perpendicular to the intersection of the two planes.

DEM. Suppose the line AB (Fig. 42.) to be the meeting of the two planes BH and AD ; and the two lines, KP in the plane AD , and LP in the plane BH , both perpendicular to the line of intersection AB . The angle KPL measured by the arc KL of a circle described from the centre P , is the same as the angle of the two planes. For, if the two planes were placed contiguous to each other, and KP coinciding with LP , and then made to diverge circularly about the line AB , the degrees of the arc LK should always mark exactly the circular motion of the two planes, as well as of the two lines LP and KL , and the arc of another circle such as LG should not; because, the circle passing through L and G should have its plane oblique to the lines LP and GP . Therefore, the arc LK is the only one of an invariable magnitude; and, of course, the angle KPL is the only invariable angle; and the true measure of the angle made by the two planes.

152. COROLL.

152. COROLL. 1. Two planes are perpendicular to each other, if a line in either plane be perpendicular to the other plane.

DEM. If the line KP in the plane AD was perpendicular to the plane AE , it should be perpendicular to all the lines in the plane AE , which pass through P (138.). Then, it should be perpendicular to the line LP , and both perpendicular to the intersection AB ; and the two planes, like the two lines, should be at right angles, and perpendicular.

2. When two planes which intersect each other are perpendicular to a third plane, the line of their intersection is also perpendicular to the third plane; and reciprocally.

DEM. By the last, each of the two has a line which is perpendicular to a line drawn in the third plane at the meeting point of their intersection. But there is only one perpendicular to the same point of a line. Therefore, the line of their intersection is that perpendicular common to both planes.

3. From the foregoing principles it will be easy to demonstrate, that a plane intersecting another plane makes with it equal vertical angles; and that a third plane intersecting two parallel planes makes with them equal corresponding, equal alternate interior, and equal alternate exterior angles. Because such angles of the planes are the same as the angles of their lines, which are perpendicular to their intersections.

4. The angle made by two plane sections through the centre of a sphere is the same as the angle which they contain of a third plane section through the centre perpendicular to them. Because the two lines of intersection of the third, with the other two, are perpendicular to the intersection of the two.

153. THEOREM

153. THEOREM 5. Three or more lines, passing through two parallel planes, and meeting at the same point, describe similar triangles or similar polygons on the two parallel planes.

DEM. Suppose the two planes PL and KS (Fig. 43.) to be parallel, five lines from the same point R describe on the two planes two similar pentagons ABDFE and *abdf**e*. For, AB and *ab* may be considered as the intersections of the same plane RAB with the two parallel planes PL and KS: therefore AB is parallel to *ab*. For the same reason, the plane RBD makes DB parallel to *db*; and the plane RAD makes AD parallel to *ad*; and the two triangles ABD and *abd* have their three sides parallel, and are similar (79.); and the angle B is equal to the angle *b*. A similar demonstration will prove the angles $E=e$, $D=d$, $F=f$, $A=a$. Therefore the five angles of the two pentagons described on the two parallel planes, by joining the intersections of the five lines from R, are equal each to each. Besides, the triangles BRD and *bRd* are similar (78.); and so are ARB and *aRb*; whence $AB : ab :: BR : bR :: BD : bd$. For the same reason, $AB : ab :: AE : ae :: EF : ef :: FD : fd$. Then, the two pentagons have their angles equal two and two, and their homologous sides proportional. Therefore, the two pentagons are similar (94.). The same demonstration may be applied to any two polygons.

154. COROL. If the two parallel planes PL and KS be produced on any side, and a pentagon equal to ABDFE be described on the prolongation of the under one, five lines from the same exterior point R, or from any other point perpendicularly equidistant from the planes, drawn to the five angles of the pentagon, will describe on the prolongation of the upper one, a
pentagon

pentagon similar to $ABDFE$, and equal to $abdfe$. The demonstration is the same as that of the last theorem, by introducing into the above series of ratios, the ratio of the perpendiculars from the point R , by means of two triangles formed by one of the five lines from R , the perpendiculars and the lines in the planes joining the perpendiculars.

SECTION

SECTION III.

OF SOLIDS.

155. THE three dimensions of extension, length, breadth, and depth, constitute a *solid*. It may be considered as formed by the motion of a surface, or as a series of contiguous surfaces its immediate elements (1.). Therefore, a solid is a space enclosed by surfaces, as a figure is a space enclosed by lines. The least number of plane surfaces requisite to enclose a space is four; and the name of such a solid is *tetraedron*. Then it is called *pentaedron*, *hexaedron*, *eptaedron*, *octaedron*, &c. as it consists of 5, 6, 7, 8, &c. outside surfaces: *polyedron* is their common name. An infinite number of infinitely small plane surfaces inclined to one another make a curvilinear solid, such as a sphere, an ellipsoid, &c. The outside plane surfaces of a polyedron make solid angles at their meetings, such as the vertex of a pyramid. The simplest solid angle requires three plane angles; and the sum of the plane angles forming a solid angle is always less than 360° .

156. A solid is said to be *regular*, when its outside plane surfaces are equal regular polygons.

Two solids are similar to each other, when they are terminated by an equal number of polygons similar two and two.

157. A

157. A polyedron takes the name of *prism*, when its superior and inferior bases are two equal polygons, and its lateral surfaces are parallelograms between the two corresponding sides of the two bases: If the two bases are two circles, it is called *cylinder*.

The length of a prism is the common meeting of two lateral parallelograms, which is a straight line (148.) or any line parallel to it between the two bases. The length of a cylinder is a straight line from its superior to its inferior circular base. The *altitude* of a prism or cylinder is a line perpendicular to the two planes of the two parallel bases. A prism or cylinder is *right* or *oblique* according as its altitude and length are parallel or oblique.

158. A prism is called *parallelepiped* if it be a hexaedron terminated by six plane parallelograms parallel two and two. It is right-angled, if the six parallelograms be rectangles, and its bases squares; and it is called *cube*, if the six planes be squares.

159. A *pyramid* is a polyedron, the base of which is a polygon, and the lateral surfaces are as many plane triangles as there are sides in the polygon of the base, meeting at a common vertex, where they form a solid angle. The pyramid is triangular, quadrangular, pentagonal, &c. according as the polygon of its base is a triangle, a quadrilatre, a pentagon, &c. If the base of the pyramid be a circle, it takes the name of *cone*.

A pyramid is *regular* whose base is a regular polygon, and whose lateral triangles are equal and isosceles. A line from the vertex perpendicular to the base of any of the lateral isosceles triangles is the *apothem* of the pyramid or of the cone. The *altitude* is a line from the vertex perpendicular to the plane of the base. In a pyramid whose base is a regular or symmetric polygon,

ELEMENTS OF

polygon, the line from the vertex to the centre of its base, is its *axis*; and a pyramid or cone is right or oblique, as the axis is, or is not perpendicular to the centre of its base.

160. A *sphere* is a solid circular in every direction, having all the points of its surface equidistant from its centre, and all its plane sections circular.

All the solids which can be the object of Geometry, are, the prism, the cylinder, the pyramid, the cone, and the sphere; of which the absolute and relative surfaces, and the solidities or volumes are to be measured.

Surfaces of Solids.

161. THEOREM I. The lateral surface of a prism or cylinder, either right or oblique, is measured by the product of its length multiplied by the perimeter of its perpendicular section.

DEM. The lateral surface of a prism is the sum of the surfaces of its lateral parallelograms. The surface of a parallelogram is the product of one of its sides by the perpendicular from the opposite one (115.). The lateral parallelograms of a prism have all an equal dimension of surface, the length of the parallelogram; and the perimeter of a section perpendicular to its length is the sum of the perpendiculars to their length (152. 2.). Therefore the lateral surface of a prism is the product, &c. The same for the cylinder, as it is a prism of an infinite number of lateral parallelograms, its perpendicular section being a circle.

162. COROLL.

162. COROLL. 1. The lateral surface of a right prism or cylinder is the product of its altitude multiplied by the perimeter of its base. Because its altitude is its length (157.) which is perpendicular to the plane, and, of course, to the sides of its base.

2. To have the total surface of a prism, or of a cylinder, the sum of the surfaces of its two bases, or the double of one of the bases is to be added to the lateral surface.

3. The total surface of a cube is six times the square of one of its dimensions. Because it consists of six equal square surfaces (158.).

4. The lateral surface of an equilateral cylinder is to the surface of its circular base as 4 to 1.

DEM. An equilateral cylinder has its altitude or length equal to the diameter of its circular base. Calling that diameter d , its lateral surface S , the surface of its base s , and the circumference of the base C ; we have, by the last, $S = Cd$, and $s = \frac{1}{4}Cd$ (123.); then $S : s :: Cd : \frac{1}{4}Cd :: 4 : 1$.

5. The total surface of an equilateral cylinder is then to the surface of its base as 6 to 1.

163. THEOREM 2. The lateral surface of a right regular *pyramid*, or of a right *cone*, is measured by half the product of the perimeter of its base by the apothem.

DEM. The lateral surface of a pyramid is the sum of the surfaces of all its lateral triangles. These are all equal (159.), and have an equal altitude, their apothem, in the right regular pyramid; the sum of their bases is the perimeter of the base of the pyramid; therefore, the sum of their surfaces is the half product of that perimeter by the apothem (116.). The same for the right cone, which is a regular pyramid of a circular base.

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164. COROLL.

164. COROLL. 1. If the pyramid was not a right regular one, the lateral triangles, not having an equal apothem, should be measured separately, and added.

2. To have the total surface of a pyramid or cone, the surface of the base is to be added to the lateral surface.

3. If a right regular pyramid or a right cone, and a right prism or cylinder have the same or equal bases, and the apothem of the first equal to the altitude of the second; their surfaces are as 1 to 2.

4. The lateral surface of an equilateral cone is to the surface of its circular base as 2 to 1.

5. The lateral surface of an equilateral cone is to the surface of its circular base as 2 to 1; and, of course, its total surface is triple of the surface of its base.

DEM. An equilateral cone has its apothem equal to the diameter of its circular base. Calling then that diameter d , the circumference of its base C , the lateral surface S , the surface of the circle of the base s ; we have $S = C \times \frac{1}{2}d$ (163.), and $s = \frac{1}{2}C \times \frac{1}{2}d$ (122.). Whence $S : s :: 2 : 1$; and $S + s : s :: 3 : 1$.

165. THEOREM 3. If a right regular pyramid, or a right cone be truncated by a section parallel to the base, its lateral surface is measured by the product of the perimeter of its middle section, multiplied by the remainder of the apothem.

DEM. The pyramid being truncated by a section parallel to the base, the perimeter of that section is a polygon similar to the polygon of the base (150.), and the corresponding sides of the two polygons are parallel (149.). Then the lateral triangles of the entire pyramid become as many trapeziums. The surface of each trapezium is the product of its middle element by its

its altitude (119.): the altitude is the remainder of the apothem, and equal in every trapezium; and the sum of their middle elements is the perimeter of the middle section between the two bases of the truncated pyramid. Therefore, the lateral surface is the product, &c. The same for a right truncated cone, which is a right regular truncated pyramid.

The same may be demonstrated by the arithmetical progression of the elements of the lateral surface of the truncated pyramid or cone.

166. THEOREM 4. The surface of a *sphere* is equal to the lateral surface of the equilateral circumscribed cylinder.

DEM. Let EG (Fig. 44.) be a square circumscribed about the circle AQB; and the diameter AB parallel to FG. If the figure revolves about the immoveable axis AB, the side FG of the square will describe an equilateral cylinder circumscribed about a sphere described by the half circle AQB. By the extremities of an infinitely small arc OS draw two parallel lines PT, KR perpendicular to the axis of rotation AB. Then, RT will describe a cylindrical ring on a circular base of the radius $PT=AC$; and the arc OS, considered as a straight line, will describe a truncated cone, whose middle element of surface is the circle described by the point D, equidistant from O and S, on the radius MD. Draw also the radius CD, and from O the line OL perpendicular to PT.

Now, the two triangles CMD, OLS, which are similar, on account of their homologous sides being perpendicular (80.) give the proportion $OS : OL = RT :: CD = PT : MD$. Call a and b the circumferences described by T and D on the radii PT and MD; and $OS : RT :: PT : MD :: a : b$; whence $OS \times b = RT \times a$.
The

The first is the lateral surface of the truncated cone described by OS (165.); the second is the lateral surface of the cylindrical ring described by RT (161.). A similar demonstration will shew the equality of every cylindrical ring described by every part of FG, and of the truncated cone described by the corresponding infinitely small arc of the semi-circle AQB. Therefore, the lateral surface of the circumscribed cylinder, and the total surface of the sphere have an equal number of equal elements of surface; and their surfaces are equal.

167. COROLL. 1. The measure of the surface of a sphere is the product of its axis AB multiplied by the circumference of its great circle (a section through the centre of the sphere); for such is the lateral surface of the circumscribed cylinder, viz. the circle on the radius BG = CD multiplied by its altitude FG = AB (162.).

2. The surface of a sphere is quadruple of the surface of its great circle. The first is, by the last, $AQBV \times AB$: the second is, $AQBV \times \frac{1}{4}AB$ (122.). Whence the surface of a circle, whose diameter is 1, being $\frac{3,14159}{4}$ (123.), the surface of a sphere of a diameter 1 is 3,14159.

3. The convex surface of a segment, or of a zone of a sphere, is measured by the product of the circumference of the great circle of the sphere multiplied by its perpendicular altitude.

DEM. The surface of the sphere has been measured (166.) by the sum of the lateral surfaces of all the truncated cones described by the infinitely small arcs, such as OS, of the generator semi-circle. Such truncated cones have a constant factor, the great circle of the sphere; and a variable factor, their altitude KP.

Therefore

Therefore any sum of such conical elements of surface making a segment or a zone will be the great circle of the sphere multiplied by the sum of their altitudes, which is the altitude of the segment, or of the zone. Hence,

The convex surface of a segment described by an arc of 60° , such as AS, originating at the extremity of the diameter of rotation AB, is equal to the convex surface of a zone, described by an arc of 30° , such as Qd, originating at the diameter VQ perpendicular to the diameter of rotation. Because they have an equal altitude $AP = cz$, the sides opposite an angle of 30° of two right-angled triangles APS and ezd having their hypotenuses equal, one a chord AS of 60° , the other the radius Cd (70.).

4. The total surface of the circumscribed cylinder is to the surface of the inscribed sphere as 6 to 4 as 3 to 2; since the first is 6, the second 4 great circles of the sphere.

5. The surface of a sphere is to the total surface of a circumscribed equilateral cone as 4 to 9.

DEM. Suppose the (fig. 17.) to revolve about the immoveable axis EA; the circle LMN describes a sphere; and the equilateral triangle EBG circumscribed to the circle, describes an equilateral cone circumscribed about the sphere. Then, the surface of the sphere is quadruple of the surface of the plane circle on the radius CL (167. 2.); and the total surface of the cone is the triple surface of the plane circle on the radius BL (164. 4.). But $CL : BC :: 1 : 2$ (71.), and $LB^2 = BC^2 - CL^2 = 2^2 - 1^2 = 3$ (90. 3.), of course $CL^2 : LB^2 :: 1 : 3$; and the surface of the sphere is to the surface of the cone as 4×1 is to $3 \times 3 :: 4 : 9$.

6. The surface of a sphere is to the surface of an inscribed regular hexagonal spheroid as 2 to the $\sqrt{3}$.

DEM.

DEM. If the (fig. 17.) revolves about the immovable diameter EA, the circle EGA describes a sphere, and the inscribed regular hexagon EFGA describes a spheroid. The surface of that hexagonal spheroid consists of a cylinder described by the side FG parallel to the diameter of rotation EA, on the circular base of the radius LG: and of two equal cones described by the two sides EF and AG, upon an equal circular base of an equal radius LG, with their apothems EF and AG both equal to FG the altitude of the cylinder. Wherefore, the sum of the lateral surfaces of the two cones is equal to the lateral surface of the cylinder (163.); and the total surface of the spheroid is twice the lateral surface of the cylinder. Now, the double lateral surface of the cylinder is $2 \times FG = EA$ multiplied by the circumference on the radius LG (162.); and the surface of the sphere is EA multiplied by the circumference on the radius CA = AG (167.). The circle on AG is to the circle on LG as AG is to LG (98.) as 2 is to $\sqrt{3}$ (from the last). Then the surface of the sphere is to the surface of the spheroid as $EA \times 2$ is to $EA \times \sqrt{3}$, as 2 to $\sqrt{3}$. Whence,

The difference between the surfaces of the sphere and of the inscribed regular hexagonal spheroid is $2 - \sqrt{3} = 2 - 1,732051$ (109.) $= 0,267949 = 0,27$ nearly.

168. THEOREM 5. The surfaces of similar solids are proportional to the squares of their homologous dimensions.

DEM. The surface of every solid is the product of two dimensions, as is obvious from the foregoing theorems. Therefore, the surfaces of similar solids are demonstrated to be as the squares of their homologous dimensions, by the same means as the surfaces of similar figures (132.).

169. COROLL.

169. COROLL. 1. The surfaces of two solids are in the compound ratio of the two dimensions, the product of which measures their surfaces; the same as (131.) $S : s :: AB : ab$. And if $A = a$, $S : s :: B : b$; if $B = b$, $S : s :: A : a$; and if $S = s$, $A : a :: b : B$; and reciprocally.

2. Because the similar solids inscribed and circumscribed to the same circle have their homologous sides proportional; the surfaces of the inscribed cylinder and equilateral cone, and of the circumscribed hexagonal spheroid may be estimated from the duplicate ratio of their dimensions with the homologous dimensions of the circumscribed cylinder and cone, and of the incircumscribed hexagonal spheroid stated in the last Corollories. It will be easy to find the duplicate ratio of the altitudes of the inscribed and circumscribed cylinders to be as 1 to 2; of the inscribed and circumscribed cones as 1 to 4; and the duplicate ratio of the right radii of the inscribed and circumscribed regular hexagonal spheroids as 3 to 4.

3. And because spheres are similar solids, being regular spheroids of an infinite number of sides; the surfaces of two spheres are as the squares of their radii, their diameters, &c. Then, as the surface s of a sphere of a diameter $d = 1$, is 3,14159 (167. 2), the surface S of another sphere of a diameter D will be found by the proportion $d^2 = 1 : D^2 :: s = 3,14159 : S = D^2 \times 3,14159$. For instance, the axis of the earth being 7912 English miles nearly, (one mile is eight furlongs = 320 poles or rods = 1760 yards = 5280 feet) the whole surface of earth, in the supposition of sphericity, is $7912^2 \times 3,14159 = 196662630$ square miles nearly.

Measure

Measure of Solidities.

170. By *solidity* geometers understand the quantity of homogeneous matter which is or may be contained within the limits or surface of a body. Therefore, it is nothing else but the contents or volume of a body, or the space it fills. The common method of measuring the solidity of a body is, to determine how many times a given unit of the same kind is contained in it. The unit used in practical geometry is a cube of a determinate magnitude, such as a cube-inch, foot, yard, mile, &c. In abstract geometry the solidity of a body may be measured by the sum of all its immediate elements of solidity, which are contiguous surfaces of an equal infinitely small thickness (1.).

Solidity of the Prism and Cylinder.

171. THEOREM 1. The solidity of a prism or cylinder, either right or oblique is measured by the product of the surface of its base multiplied by its altitude.

DEM. Suppose T (Fig. 45.) to be a small cube agreed upon as a unit of solidity; and that one of its square surfaces is contained in the surface of the base ABCDE of the right prism Aa, a number of times n . If a stratum be taken off the right prism, by a section parallel

parallel to its base, and of the same altitude as the standard T , that stratum will contain a number n of small cubes equal to T . Therefore, if the altitude of the right prism contains the altitude of the standard a number of times s , it will have a number s of strata, each of which contains a number n of small cubes equal to T . Whence, the number of such small cubes contained in the whole right prism is $n \times s$, the product of the surface of its base by its altitude.

Again. The elements of solidity of the right prism are surfaces equal to its base $ABCDE$, as many as there are points in its altitude Aa ; and their sum, which is the solidity of the prism, is the surface of the base multiplied by the altitude of the prism.

The solidity of the oblique prism $ABCDEL$ is also measured by the product of the surface of its base multiplied by its altitude. For it consists of elements of solidity equal to its base (150.) and contains as many of such elements as there are points in its altitude. So that, if the right and oblique prisms on the same base $ABCDE$ are between parallels, they have an equal solidity.

The solidity of an oblique prism may also be measured by the product of the surface of a plane section perpendicular to its length multiplied by its length. The demonstration will be easily made by taking the corresponding elements of solidity in the right and oblique prisms lengthwise, and shewing the reciprocal proportion of their dimensions.

The same may be said of a cylinder: because it is a prism of a polygonal base of an infinite number of sides.

172. COROLL. 1. Any two prisms or cylinders have their solidities in the compound ratio of the surfaces of their bases, and their altitudes. For, their

N

solidities

solidities being expressed by S and s , the surfaces of their bases, by B and b ; their altitudes, by A and a ; $S = AB$, and $s = ab$; therefore, $S : s :: AB : ab$. Whence, if $A = a$, $S : s :: B : b$; if $B = b$, $S : s :: A : a$; if their bases are similar polygons, and two of their homologous dimensions are D and d ; $S : s :: A \times D^2 : a \times d^2$; and if $S = s$, $A : a :: b : B$.

2. The solidity of a cube is equal to the third power of its dimension. For the altitude of a cube is equal to the side of its square base. Therefore, a cube being a prism, its contents is the square of its dimension multiplied by the dimension itself, which is the cube of the dimension. Hence the denomination of cube to the third power of any quantity.

3. The geometrical cube of a binom may be easily demonstrated to consist of the same parts as the arithmetical cube, viz. the cube of both terms, and the triple square of each multiplied by the other. The same might be done for the cube of any polynom, as has been done for the square (115. 4.).

173. THEOREM 2. A prism may be divided into as many triangular prisms as there are sides in the polygon of its base less two, all of the same altitude.

DEM. Diagonals from the homologous angles A , a , L , (Fig. 45.) of the under and upper bases of the prisms to the opposite angles, divide the equal polygons of the bases into as many triangles as there are sides less two (67.); and the corresponding triangles are equal, $ADE = ade = LFM$, &c. (150.). A plane section in the plane of the parallelogram $aADd$ or $LFDA$ cuts off a triangular prism in each, having the inferior and superior bases equal triangles $ADE = ade = LFM$. A plane section in the plane of the parallelogram $aACc$ cuts off a triangular prism having the bases ABC and abc , or LGO ;

LGO; and leaves another triangular prism, the bases of which are ACD and *acd* or LOF. The altitudes of these triangular prisms are equal, viz. the perpendiculars between the planes of the parallel bases ABCDE and *abcde'* or LGOFM.

Solidity of a Pyramid.

174. THEOREM 1. The plane sections or elements of solidity parallel to the plane base of a pyramid are polygons similar to the polygon of its base.

DEM. Straight lines from the vertex of a pyramid to each of the angles of the polygon of its base are lines from the same point R (Fig. 43.) to the points ABDFE of the plane LP, which describe similar figures such as *abdf'e'*, on any plane KS parallel to the plane LP (153.). So, ABDFER may be taken for a pyramid.

175. COROLL. 1. The elements of solidity of a pyramid decrease from the base to the vertex in the duplicate ratio of their distances from the vertex.

DEM. Call the surface ABDFE, S; and the surface *abdf'e'*, s. As they are similar, $S : s :: \overline{AB}^2 : \overline{ab}^2$ (132.) $:: RA^2 : Ra^2$, by the similar triangles ARB and aRb. If a line be drawn from R perpendicular to the two planes KS and LP, and that perpendicular joined by two lines in the planes, one from A, the other from a, these two lines will make two similar triangles with AR and aR, in which RA is to Ra as the perpendicular from R to the plane LP is to the perpendicular from R to the plane KS. Therefore S is to s as the square of the

the perpendicular or distance from the vertex R to the plane EB is to the square of the perpendicular or distance from R to the plane eb .

2. The successive elements of the solidity of a pyramid increase from the vertex to the base as the squares of the successive terms of an arithmetical progression of an infinite number of terms.

DEM. The successive plane sections parallel to the base, which are the elements of solidity correspond to the successive geometrical points of the perpendicular from the vertex to the plane of the base. The distances from the vertex to those successive points are in an arithmetical progression; and the successive elements of solidity are as the squares of these distances.

3. The corresponding elements of solidity are equal in pyramids, either right or oblique, which have the same or an equal base and an equal altitude.

DEM. Corresponding elements of solidity are plane sections parallel to the base and at equal distances from the vertex. If then the equal or common base of two pyramids be S , any two corresponding elements s and s' ; the altitude of S from the vertex, D ; and of s and s' , d ; we have $S : s :: D^2 : d^2$; and $S : s' :: D^2 : d'^2$; whence evidently $s = s'$.

4. Two pyramids either right or oblique, which have equal bases and equal altitudes have also their solidities equal.

DEM. Two such pyramids have the same number of equal elements of solidity, since their corresponding elements are equal, and their number is equal to their altitude. Therefore, their sum, which is the solidity of the pyramids, is the same in both.

5. Because

5. Because a cone is a pyramid, whose base is a polygon of an infinite number of sides, the last theorem and corollories may be applied to cones.

176. THEOREM 2. Any triangular prism may be divided into three pyramids of an equal solidity.

DEM. Suppose the (Fig. 46.) to be a triangular prism having the two equal and parallel triangular bases ABD and FEG, and the three lateral parallelograms FD, BG, BF. Draw the two diagonals EA and ED of the two lateral parallelograms BF and BG, and make a plane section by the plane of the triangle AED; that section cuts off from the triangular prism a triangular pyramid of the base ABD having its vertex at E; and leaves a quadrangular pyramid of the base ADGF having its vertex at E. Draw on that quadrangular base the diagonal AG, and make a plane section by the triangle AEG; it will divide the quadrangular pyramid into two equal triangular pyramids, one of the base AGD, the other of the base AFG $\equiv$ AGD (61.) both having their vertex at E (175. 4.). But the last one on the base AFG having its vertex at E may be considered as being on the base FEG, and having its vertex at A, and, of course, of the same solidity as the first on the base ABD having its vertex at E. Therefore the three pyramids ABDE, FEAG, AGDE, into which the triangular prism has been divided are of an equal solidity.

177. COROL. Any triangular pyramid, such as ABDE, may be completed into a triangular prism of a triple solidity, by the addition of two pyramids having each a solidity equal to its own. Of course, the solidity of a triangular pyramid is the third part of the solidity of a triangular prism of the same base and altitude.

178. THEOREM 3. A pyramid may be divided into as many triangular pyramids as there are sides in
the

the polygon of its base less two, all of the same altitude.

DEM. From the angle A (Fig. 43.) of the base of the pentagonal pyramid, draw diagonals AD, AF to the opposite angles. Then make two plane sections from the vertex R by the plane triangles ARD, ARF; and the pentagonal pyramid is divided into three triangular pyramids ABDR, AEFR, AFDR, with a common vertex R and their bases in the same plane, of course, of the same altitude, the perpendicular from R to the plane of the base. The same demonstration is applicable to any pyramid, whatever be the polygon of its base.

179. COROL. The solidity or volume of a pyramid, either right or oblique, whatever be the polygon of its base, is the third part of the volume of a prism of the same base and altitude.

DEM. Every one of the triangular pyramids, into which the pentagonal one has been divided in the last theorem, may be completed into a triangular prism of the same base and altitude, and of a triple solidity (177.). The three triangular prisms joined together make a pentagonal prism of the same base and altitude as the pentagonal pyramid. Therefore, the pentagonal pyramid may be completed into a pentagonal prism of the same base and altitude, and of a triple solidity.

And because a cone is a pyramid; the solidity of a cone is the third part of the solidity of a cylinder of the same base and altitude.

180. THEOREM 4. The solidity of a pyramid or cone, either right or oblique, is measured by the third part of the product of the surface of its base, multiplied by its altitude.

DEM.

DEM. The solidity of a pyramid or cone, is, by the last, the third part of the solidity of a prism or cylinder of the same base and altitude. But the solidity of a prism or cylinder, either right or oblique, is the product of the surface of its base multiplied by its altitude (171.). Therefore, &c.

Again. The successive elements of the solidity of a pyramid increase from the vertex to the base as the squares of the successive terms of an arithmetical progression of an infinite number of terms (175. 2.); analysis has demonstrated that the sum of all the terms of such a progression of squares is the product of the last by the third part of their number. The surface of the base is the last term; and the altitude of the pyramid is the number of terms. Therefore, analysis gives the same measure of the solidity of the pyramid as geometry.

181. COROL. If a pyramid be truncated by a plane section parallel to its base, the solidity of the remaining frustum, or truncated pyramid is obviously the difference of the solidities of the entire pyramid and of the small one cut off. To calculate it; call the surface of the base ABDFE (Fig. 43.) S ; the surface of the plane section parallel to the base $abdf'e'$, s ; the perpendicular altitude of the frustum, A ; the perpendicular altitude of the second pyramid, a ; which gives the altitude of the entire pyramid $A + a$: and call the solidity of the frustum V . Then $V = \frac{1}{3}S \times (A + a) - \frac{1}{3}s \times a$. In which expression a the only quantity unknown is found by $AB : ab :: A + a : a$ (78.); whence $AB - ab : ab :: A + a - a = A : a = \frac{ab \times A}{AB - ab}$.

Here is another solution of the same by analysis, with the same denominations as in the last, except S^2 for the surface of the base of the pyramid, and s^2 for the surface

surface of the plane parallel section. $S^2 : s^2 :: (A + a)^2 : a^2$ (132.), and $S : s :: A + a : a$; whence $S - s : s :: A : a$;

and $a = \frac{As}{S-s}$; $A = \frac{aS-as}{s} = \frac{As}{s}$. Then, $A + a =$

$$\frac{As}{S-s} + \frac{As}{s} = \frac{As}{S-s} + \frac{As \times S-s}{s \times S-s} = \frac{As + AS - As}{S-s} = \frac{AS}{S-s}.$$

Now, the volume of the frustum $V = \frac{1}{3}S^2 \times \frac{AS}{S-s} -$

$$\frac{1}{3}s^2 \times \frac{As}{S-s} = \frac{1}{3}A \times \frac{S^3-s^3}{S-s} = \frac{1}{3}A \times (S^2 + Ss + s^2). \quad \text{That is,}$$

the solidity of the truncated pyramid is measured by the $\frac{1}{3}$ part of its altitude A multiplied by the sum of the surfaces of the two bases $S^2 + s^2$ and of the mean geometrical proportional Ss between the surfaces of the superior and inferior bases.....The same of a truncated conc.

Solidity of a Sphere.

182. THEOREM 1. The solidity of a sphere is the $\frac{2}{3}$ part of the solidity of its circumscribed cylinder.

DEM. About the circle APB (Fig. 47.) circumscribe the square FM. Draw the two diameters AB, NP, perpendicular to each other, and parallel to the sides DF and FE; also the two oblique radii CF, CD, making two right-angled isosceles triangles CAF, CAD; the line TL perpendicular to AB, and the radius CO = CA = AF = KL. Suppose the whole figure to revolve about the immoveable diameter AB. By the rotation, the half circle APB describes a sphere; FE, a circumscribed cylinder, on a base equal to the circle; the arc AP

AP of 90° , a hemisphere; FP, the corresponding half cylinder; FC, a cone of the same base and altitude as the half cylinder; and the three lines KR, KO, KL, three circles on the centre K, the corresponding elements of solidity of the cone, the sphere, and the cylinder.

Now, in the right-angled triangle CKO, $CO^2 = KL^2 = KO^2 + KC^2 = KO^2 + KR^2$ (89.); because $KR = KC$, as $AF = AC$. Therefore, the surface of the circle on the radius KL, which is the element of solidity of the cylinder, is equal to the sum of the surfaces of the two circles on the radii KO, KR, which are the corresponding elements of solidity of the sphere and of the cone (135. 1.). The same demonstration takes place for any other parallel situation of the line KROL from A to C. Thence the sum of all the elements of solidity, or the solidity itself of the half cylinder, is equal to the sum of all the elements of solidity, or to the sum of the solidities of the hemisphere and the cone. But the solidity of the cone is the third part of the solidity of the half cylinder (179.). Then the solidity of the hemisphere is the $\frac{2}{3}$ part remaining of the solidity of the half cylinder. Therefore, the solidity of the entire sphere is the $\frac{2}{3}$ part of the solidity of the entire circumscribed cylinder.

183. THEOREM 2. The solidity of a sphere is measured by the product of its convex surface multiplied by the $\frac{1}{3}$ part of its radius.

DEM. The surface of the plane great circle of a sphere is equal to the surface of the circular base of its circumscribed cylinder; and the altitude of the cylinder is the axis or double radius of the sphere. Taking then C for the surface of the great circle, and R for the radius; the solidity of the circumscribed cylinder is $C \times 2R = 2CR$ (171.). By the last, the solidity of

of the sphere is $\frac{2}{3}2CR = \frac{4CR}{3} = 4C \times \frac{R}{3}$. But $4C$ is the convex surface of the sphere (167. 2.). Therefore, &c.

Again. The convex surface of the sphere may be considered as consisting of an infinite number of infinitely small plane polygons, such as fill exactly a space (72. 4.). Then, lines from every angle of such polygons to the centre of the sphere will divide the whole sphere into as many pyramids as there are polygons: every pyramid will have its vertex at the centre of the sphere, and the same altitude, the radius of the sphere. The sum of their bases is the convex surface. Therefore, the sum of the solidities of all the pyramids, which is the solidity of the sphere, is the convex surface of the sphere multiplied by the $\frac{1}{3}$ part of the radius (180.).

184. COROLL. 1. As the convex surface of a sphere, whose axis is 1, is 3,14159 (167. 2), and as the axis is double of the radius; its solidity is $\frac{1}{2} \times 3,14159 = 0,523598$.

2. The solidities of a right cone of a base equal to the great circle of the sphere, and an altitude equal to its axis; of the sphere; and of a circumscribed cylinder, are as 1, 2, 3, or as 2, 4, 6. For, the great circle of the sphere being expressed by C , the radius

by R ; the solidity of the cone is $\frac{2CR}{3}$ (180.); the soli-

dity of the sphere is $\frac{4CR}{3}$ (183.); the solidity of the cylinder $2CR$ (171.). Then, the solidity of the sphere is middle arithmetical proportional between the solidities of the cone and of the cylinder.

3. The solidity of the inscribed equilateral cone is to the solidity of the sphere as $1\frac{1}{8}$ to 4.

DEM.

DEM. The equilateral inscribed triangle BEG (Fig. 17.) describes the inscribed equilateral cone by the rotation of the figure about EA, on a circular base of the radius LB, and the altitude $EL = \frac{3}{2}R$ (71.); whence, the surface of its circular base being expressed by G, and the radius of the sphere by R, the surface of its great circle by C, its solidity is $\frac{1}{2}GR$; and the solidity of the sphere is $\frac{4}{3}CR$. But $G : C :: 3 : 4$, then, $\frac{1}{2}GR : \frac{4}{3}CR :: \frac{3}{2}R : \frac{16}{3}R :: 9 : 32 : 1\frac{1}{8} : 4$.

4. The solidity of a hexagonal inscribed regular spheroid is to the solidity of the sphere as 3 to 4.

DEM. With the same denominations as in the last, the solidity of the inscribed regular hexagonal spheroid described by the rotation of (Fig. 17.), consists of a cylinder whose solidity is GR; and of two cones, whose solidity is $\frac{1}{6}GR + \frac{1}{6}GR = \frac{1}{3}GR$. The solidity of the spheroid is then to the solidity of the sphere as $GR + \frac{1}{3}GR = \frac{4}{3}GR$ to $\frac{4}{3}CR :: \frac{12}{3}R : \frac{16}{3}R :: 3 : 4$.

Resuming the three last corollories, we find the solidities, of a cone inscribed in a sphere, of a cone inscribed in the circumscribed cylinder, of the inscribed regular hexagonal spheroid, of the sphere, and of the circumscribed cylinder, to be as $1\frac{1}{8}$, 2, 3, 4, 6.

5. The solidity of a spherical stratum, contained between two concentric and homogeneous spheres is equal to the difference of the solidities of the two concentric spheres. From this may be calculated the weight of the atmosphere which surrounds the earth. We know, by the barometer, that a column of air is of the same weight as a pillar of mercury about 30 inches, or a pillar of water about 36 feet high. The cubic foot of water weighs $62\frac{1}{2}$ pounds avoirdupois, of course a column of air of a square foot base weighs as much as
a pillar

a pillar of water of 36 feet, or a pillar of mercury of 30 inches, viz. $62\frac{1}{2} \times 36 = 2250$ pounds. Now, as 30 inches, or even 36 feet do not sensibly increase the radius of the earth, 2250 pounds multiplied by the number of square feet in the convex surface of the earth, as stated (169. 3.), will give the total weight of the atmosphere.

6. The solidity of a spherical sector, such as the one described by the circular sector OCS (Fig. 47.), in the rotation of the figure about AB, is measured by its convex surface multiplied by the one-third part of the radius. For, the solidity of the sector may be considered as consisting of a number of pyramids having their vertex at the centre, with a common altitude, the radius; and the sum of their bases forming its convex surface (183.).

7. The solidity of a segment, such as the one described by the rotation of the arc OAS above the chord OS, is the $\frac{2}{3}$ parts of the solidity of a cylinder of the same thickness AK; that is, the product of the surface of the great circle of the sphere multiplied by its thickness AK (182.).

Geometry does not measure the solidity of irregular bodies. Hydrostatics afford easy means for such measurements, from the specific gravities of bodies.

Relations

Relations of Solidities.

THE foregoing measurement of solidities is the absolute estimate of the contents or volumes of bodies. The relation of solidities is their relative or proportional estimate.

185. THEOREM 1. The solidities of two bodies are in the compound ratio of three of their dimensions.

DEM. From the foregoing theorems, the solidity of a body is always the product of a surface multiplied by another dimension. A surface is always the product of two dimensions. Therefore, expressing the solidities of two bodies by S and s ; the two dimensions of the surfaces taken in the measure of their solidities by A and a , by B and b ; the third dimension by H and h ; we have $S : s :: ABH : abh$; the compound ratio of $A : a$; $B : b$; $H : h$.

186. COROL. Then if two solids have an equal dimension, their solidities are as the product of the other two dimensions. If $A=b$, $S : s :: BH : ah$. If $AH=ab$, $S : s :: B : h$. If $AB : ab :: h : H$; $S=s$, &c.

187. THEOREM 2. The solidities of similar solids are proportional to the cubes of their homologous dimensions.

DEM. The homologous dimensions of similar solids are proportional (156.). So, keeping the same denominations as in the last, $A : a :: B : b :: H : h$. Thence, the above compound ratio of the three dimensions consists of three equal ratios; and is the same as
of

of the cubes of the simple ratios ; that is, $S : s :: ABH : abh :: A^3 : a^3 :: B^3 : b^3 :: H^3 : h^3$.

188. COROLL. 1. The solidities of two similar prisms or cylinders, of two similar pyramids or cones, are as the cubes of the similar dimensions of their bases, of their altitudes, &c. The solidities of two spheres are as the cubes of their radii, of their axis, &c.

2. Having determined (184.) the ratio of the solidity of a sphere to the solidities, of an inscribed cone, of a regular hexagonal inscribed spheroid, of a circumscribed cylinder, and its inscribed cone ; it will be easy to find the ratio of the solidity of the sphere to the solidities, of a circumscribed cone, of a regular circumscribed hexagonal spheroid, of an inscribed cylinder, from the ratios of the cubes of their homologous dimensions.

3. The solidity of a sphere of an axis equal to 1 having been found to be 0,523598 (184.) ; taking $d = 1$, $s = 0,523598$, and D for the axis of another sphere, S for its solidity ; S will be found by the following proportion $d^3 : D^3 :: 0,523598 : S = D^3 \times 0,523598$. Hence the axis of the globe of the earth being 7912 English miles, its solidity is $7912^3 \times 0,523598 = 259332421204$ English cubic miles, nearly. Which may be reduced to furlongs, rods, yards, feet, &c.

REMARK. The foregoing absolute and relative measurement of solidity gives only the contents or volumes of bodies. The weight of homogeneous matter is calculated in hydrostatics, by the specific gravities of material substances compared to distilled water, which weighs 1000 ounces (avoirdupois) under the volume of one cubic foot ; and is taken as the universal standard. For instance, the specific gravity of water

water being 1, that of pure gold of 24 caracts is 19,258 which multiplied by 1000, is the absolute weight of one cubic foot of pure gold, in ounces. Whence the weight of a cubic inch of pure gold should be 19,258 multiplied by 1000 and divided by the cube of 12, which is equal to 11,144 ounces.

From the estimate of the contents of the earth calculated in English cubic miles (188. 3.) its weight could be calculated if it was homogeneous. But as it is a compound of all the material substances, which are of such various specific gravities, nothing can be stated about its weight, even by approximation, from the geometrical principles. If, at a mean, it could be taken as a homogeneous globe, of the specific gravity of water; its weight should be 259332421204 multiplied by the cube of 5280, the number of feet contained in an English mile, and multiplied again by 1000 ounces or $62\frac{1}{2}$ pounds.

END OF THE ELEMENTS OF GEOMETRY.

TRIGONOMETRY.

190. TRIGONOMETRY, one of the most useful branches of abstract and practical geometry, is the science and art of resolving a triangle; that is, of calculating the magnitude of its angles and sides, in all the cases where the data are sufficient, and do not leave the parts of the triangle to be found of a variable and indeterminate magnitude.

Besides the rectilineal triangles, the sides of which are three straight lines in the same plane, and of which elementary geometry has given the general properties; there is another kind of triangles, made on the surface of a sphere by three arcs of its great circles, thence called *spherical* triangles. These have their special properties, which are the peculiar object of trigonometry. Thence arise two sorts of trigonometry. The resolution of plane triangles is the object of *plane* trigonometry. The resolution of spherical triangles is the object of *spherical* trigonometry.

P

Elementary

Elementary geometry resolves a triangle, in some few cases, by means of the proportionality of its sides to the sides of another similar triangle, previously measured, as has been done (92. 93.) Trigonometry resolves a triangle, in all cases, by means of the *sines*, *tangents*, and *secants* of angles and circular arcs, called trigonometrical lines.

PLANE

PLANE TRIGONOMETRY.

191. **PLANE** Trigonometry is the science and art of solving this general problem. Out of the six parts of a plane triangle, viz. Three angles and three sides, any three being given, among which one side, to find the other three?

The *data* required for the solution of the problem are the same as for the equality of triangles (55. 58.) Because any two things, even the three angles without a side, leave the remaining parts of a triangle variable and indeterminate in their magnitude.

As the Trigonometrical lines are the means of resolving both plane and spherical triangles; tables of such lines are requisite for every angle or arc. Therefore a treatise on Trigonometry requires 1°. the theory of sines, tangents and secants, 2°. the principles for the construction of tables of sines, tangents and secants; 3°. the method of applying such lines to the resolution of triangles. Examples shall be subjoined of a few solutions of the principal cases.

Theory of Sines, Tangents, and Secants.

192. **DEFINITIONS.** The *sine* (right sine) of a central angle ACB (fig. 48.) or of the arc AB, which is the measure of the angle, is a line AP from any point A of either of the two sides AC for instance, perpendicular on the other side CB; or from one extremity A of
of

of the arc perpendicular on the radius CB passing at the other extremity B of the arc.

A line BT perpendicular on the extremity of the radius CB, and terminated at the radius CA, produced is the *tangent* of the same angle ACB or arc AB; and the radius CA produced to the tangent is the *secant* of the same angle or arc.

The part BP of the radius between the arc and the sine, is the *versed sine* of the same angle or arc.

Likewise for the angle ACD or the arc AD, its sine is AS; its tangent DG; its secant CG; its versed sine DS. And if the two angles ACB and ACD, or the two arcs AB and AD are complement to each other: the sine, tangent, secant, versed-sine of either is called the *co-sine*, the *co-tangent*, the *co-secant*, the *co-versed sine* of the other. So if BCD be a right angle or if $BD = 90^\circ$; AP the sine of AB is the co-sine of AD; and AS the sine of AD is the co-sine of AB. Likewise BT is the co-tangent of AD, and DG is the co-tangent of AB, &c.

These are all the trigonometrical lines which in the calculations are generally expressed by the following abbreviations: *sin. tang. sec. v-sin. cos. cot. co-sec. co-v-sin.* The radius is always expressed by the single letter R.

193. COROLL 1. The sine, tangent, secant, and of course the co-sine, co-tangent, co-secant, of an obtuse angle ACE or of an arc $AE > 90^\circ$ are the same as of its supplement ACB or AB. Because, the lines from either of the two radii CA or CE cannot meet the other perpendicularly, unless it be produced on the opposite side of the centre.

Then $ap = AP$ is the sine of ACE; its tangent is $EM = BT$; its secant is $CM = CT$; the co-sine is AS; the co-tangent is DG; and the co-secant CG; all the same

same as for the supplement ACB or AB . But the versed sine of the obtuse angle or arc $Bp = EP = EC + CP = EC + AS$ the sum of the radius and the co-sine; while the versed sine of the acute angle ACB or arc AB is equal to $BC - CP = BC - AS$ the difference of the radius and the co-sine.

2. The sine of a right angle BCD or of an arc of 90° BD is the radius itself DC . So the arc increasing from B to D from 0° to 90° , the sine AP increases from 0 to radius; then from D to E , from 90° to 180° , the sine decreases from radius to 0. Therefore, the sine of 90° is the greatest of all, and is called the *total sine*. But the co-sine decreases from 0° , where it is the radius itself to 90° , where it vanishes.

Likewise the tangent BT , which is equal to 0 at 0° , encreases to infinity as the arc encreases from B to D , where there is no tangent as it becomes parallel to CD , and the secant, which is the radius at 0° , encreases to infinity as the arc encreases from 0° to 90° where there is no secant. Then from 90° to 180 , the tangent decreases from infinity to 0, the secant from infinity to radius. But they are both, like EN and CN , on the other side of the diameter BE ; and of course negative.

What we call 0° must be rather understood of an infinitely small arc: because, when there is no arc there is no sine, no tangent, no secant, &c.

3. In a right-angled triangle, such as APC ; if the hypotenuse AC be taken as radius, each of the small sides AP and CP is the sine of its opposite acute angle, and the other is its co-sine. AP is the sine of ACP ; and $CP = AS$ is the sine of the complement $CAP = ACD$. But if in a right-angled triangle, such as CBT , either of the small sides such as CB , is taken for radius, the other small side BT is the tangent of its opposite angle BCT ; and the hypotenuse is its secant.

194. THEOREM 1. The sine of an arc is half the cord of a double arc.

DEM. If the sine AP perpendicular to CB be produced to H, the chord $AH = 2AP = 2 \sin. AB$; and the arc $AH = 2AB$ (59. 2.)

195. COROLL. 1. The sine of an angle or arc of 30° is the half of the total sine or radius, for the sine of 30° is, by the last, half of the chord of 60° , which is equal to the radius (70.) Therefore the sines encrease as the arcs encrease from 0° to 90° ; but not in the proportion of the arcs. They encrease more rapidly for small than for great arcs. However, the sines of infinitely small arcs may be taken proportional to their arcs, without a sensible error; because they coincide sensibly with their arcs.

2. The tangent of 45° is equal to the radius or total sine. For when $AB = 45^\circ$ the right-angled triangle CBT is isoscles, and $CB = BT$ (43.) Nevertheless, tangents are not proportional to the angles or arcs; since from 45° to 90° , the tangent encreases from radius to infinity (193. 2.)

196. THEOREM 2. The cosines and secants of two arcs are reciprocally proportional; and so are the sines and co-secants; and so the tangents and co-tangents.

DEM. 1^o. For the cosines and secants. The two similar triangles CPA and CBT give the proportion $CP : CB :: CA : CT$. or $\div \cos. AB : R : \sec. AB$. Thence, for any other arc EO, the similar proportion $\div \cos. EO : R : \sec. EO$; and the product of the co-sine of any arc multiplied by its secant, is constant and equal to the square of the radius. Taking then any two arcs a and b ; $\cos. a \times \sec. a = \cos. b \times \sec. b$ and $\cos. a : \cos. b :: \sec. b : \sec. a$.

2. For the sines and co-secants. By the similar triangle CSA, CDG; $CS = AP : CA :: CD : CG$.

Whence

Whence $\sin. AB \times \operatorname{cosec}. AB = R^2$. And for the arc EO ; $\sin. EO \times \operatorname{cosec}. EO = R^2$. Thence for two arcs a and b ; $\sin. a \times \operatorname{cosec}. a = \sin. b \times \operatorname{cosec}. b$; $\sin. a : \sin. b :: \operatorname{cosec}. b : \operatorname{cosec}. a$.

3. For the tangents and co-tangents. By the similar triangles CBT, CDG ; $BT : CB :: CD : DG$. Whence for any two arcs a and b ; $\operatorname{tang}. a \times \cot. a = \operatorname{tang}. b \times \cot. b$; and $\operatorname{tang}. a : \operatorname{tang}. b :: \cot. b : \cot. a$.

197. THEOREM 3. In any triangle the sines of the angles are proportional to the opposite sides.

DEM. Every triangle may be inscribed in a circle; and then every side of the triangle is the chord of an arc, double of the arc which measures the opposite angle (30.) Therefore the sine of every angle of a triangle is equal to half the opposite side (194.) The three sides are proportional to their halves; of course, the sides are proportional to the sines of the opposite angles.

All the cases of the general problem of plane Trigonometry, are resolved by the application of this last theorem.

198. THEOREM 4. A perpendicular from any angle of a triangle to the opposite side, divides that side into two such segments, as to have the opposite side to the sum of the two other sides, as the difference of the same two sides is to the difference, or the sum of the two segments of the opposite side, (according as the perpendicular falls on the opposite side, or the opposite side produced.)

This theorem is demonstrated in geometry (103.)

199. THEOREM 5. The sum of the sines of two arcs, is to their difference, as the tangent of half the sum of the same two arcs, is to the tangent of half their difference.

DEM

DEM. Take (fig. 49.) the arc $AB=a$ and the $DB=b$, b the greater. Draw the diameter BM , and the chord AO parallel to BM ; also AP the sine of AB , and DR the sine of DB produced to G , with the chords OD , OG . Then the sine of the two arcs $AB+DB=AB+BG=AG=a+b$; and their difference $b-a=DB-AB=AD$. Also, the sum of the two sines, $\sin. a+\sin. b=AP+DR=AP+RG=RK+RG=KG$; and their difference $\sin. b-\sin. a=DR-KR=DK$.

From O as a centre, describe a circle FE through the centre C , of the same radius as the circle DGO . Draw from OD to OG the line NT perpendicular to OA , and tangent to the circle FE at the point L . Then, LT is the tangent of the arc LF , and LN is the tangent of the arc LE (192.) But the arc LF is the half of the arc AG ; since the same angle AOG is central to the circle EF , and inscribed in the circle DGO (30); and for the same reason the arc $LE=\frac{1}{2}AD$ both the measure of the same angle DOA . Therefore $LT=\text{tang.}\frac{a+b}{2}$; and $LN=\text{tang.}\frac{b-a}{2}$.

Now from the similar triangles KOG , LOT , $KG:LT::KO:LO$; and by the similar triangles KOD , LON , $DK:LN::KO:LO$. Therefore $KG:DK::LT:LN$. That is $\sin. a+\sin. b:\sin. b-\sin. a::\text{tang.}\frac{a+b}{2}:\text{tang.}\frac{b-a}{2}$. Thence

200. COROL. The sum of the cosines of two arcs is to their difference, as the cotangent of half the sum of the same two arcs is to the tangent of half their difference.

DEM. Suppose the same two arcs a and b , as in the last, b the greater; and a' to be the complement of a , and

and b' the complement of b . Then $a + a' = b + b' = 90^\circ$; whence $a + b$ and $a' + b'$ being the supplement of each other, $\frac{a+b}{2}$ and $\frac{a'+b'}{2}$ are complement to each other, and the tangent of either is the co-tangent of the other; and $b - a = a' - b'$ which gives $\text{tang. } \frac{b-a}{2} = \text{tang. } \frac{a'-b'}{2}$. as $b > a$, so $a' > b'$. Now, by the last,

$\text{Sin. } a' + \text{sin. } b' : \text{sin. } a' - \text{sin. } b' :: \text{tang. } \frac{a'+b'}{2} :$
 $\text{tang. } \frac{a'-b'}{2} = \text{tang. } \frac{b-a}{2}$ and substituting the cos. of the arcs for the sines of their complements $\cos. a + \cos. b : \cos. a - \cos. b :: \cot. \frac{a+b}{2} : \text{tang. } \frac{b-a}{2}$.

201. THEOREM 6. The sum of any two sides of a triangle is to their difference, as the tangent of half the sum of the two opposite angles, is to the tangent of half their difference.

DEM. Call the two angles a and b as above, b the greater; and their two opposite sides S and s , S the greater opposite to b . By (197) $S : s :: \text{sin. } b : \text{sin. } a$; whence $S + s : S - s :: \text{sin. } b + \text{sin. } a : \text{sin. } b - \text{sin. } a$; and substituting the ratio of the sum and difference of the sides for the sum and difference of the sines in the above proportion, (199) $S + s : S - s :: \text{tang. } \frac{a+b}{2} : \text{tang. } \frac{b-a}{2}$.

Q

Construction

Construction of the Tables of the Trigonometrical Lines.

202. THE tables of trigonometrical lines are the expressions of the sines, tangents, secants, &c. of every angle or arc by numbers. The total sine or radius of a circle is taken for their common measure. Therefore the radius is supposed to be divided into a great number of equal parts; for instance, into 10,000,000,000, or rather it is supposed to be equal to 1, and that unit divided into decimal parts of the denominator 10,000,000,000; and every trigonometrical line is expressed in parts of the radius.

With such tables, any angle or arc being given, its trigonometrical lines are found immediately; and any trigonometrical line being given, the angle or arc to which it belongs is easily found: which affords ready means of calculation for the resolution of triangles.

Such tables contain only the trigonometrical lines of the angles and arcs from 0° to 90° : because, the trigonometrical lines of an obtuse angle or arc, are the same as of its supplement, the versed sine alone excepted (193.)

The sine or tangent of an infinitely small arc, may be taken as coinciding with, and equal to the arc itself, like the chord of an infinitely small arc (109.)

The following problems contain the principles for the calculation of the trigonometrical lines and the construction of their tables; taking always the radius $R=1$.

203. PROBLEM

203. PROBLEM 1. The sine of an arc being given, to find the cosine, tangent, secant, co-tangent, co-secant, versed sine, and co-versed sine?

Suppose AP (fig. 48.) the sine of the arc AB to be given, AD being the complement of AB, the four right-angled triangles CPA, CBT, CSA, CDG are similar, and their homologous sides are proportional.

SOL. 1. In the triangle CPA, $CP^2 = CA^2 - AP^2$; that is, $\cos.^2 = R^2 - \sin.^2 = 1 - \sin.^2$; and $\cos. = \sqrt{1 - \sin.^2}$

2. By CPA and CBT; $CP : CB :: AP : BT$; that is, $\cos. : R :: \sin. \text{ tang.} = \frac{\sin.}{\cos.}$ and $CP : CB ::$

$CA : CT$; that is, $\cos. : R :: R : \sec. = \frac{1}{\cos.}$

3. By CBT, CDG; $BT : CB :: CD : DG$; that is, $\text{tang.} : R :: R : \cot. = \frac{1}{\text{tang.}}$ or by APC, CDG; $AP : CP :: CD : DG$; that is, $\sin. : \cos. :: R : \cot. = \frac{\cos.}{\sin.}$

4. By CPA, CDG; $AP : CA :: CD : CG$; that is, $\sin. : R :: R : \text{cosec.} = \frac{1}{\sin.}$

5. For an acute or obtuse angle, $v\text{-sin.} = 1 \mp \cos.$

204. PROBLEM 2. The sine and cos. of an arc a , being given, to calculate the sine of half the same arc $\frac{1}{2}a$?

SOL. Call any arc AB, a , (fig. 50), and let its sine AP and its cos. CP be given. Draw the chord AB, and the radius CD perpendicular on it, bisecting the chord and arc AB. Then is AL the sine of half the arc AB; to find which, take in the right-angled triangle APB, $AB^2 = AP^2 + BP^2 = \sin.^2 a + v\text{-sin.}^2 a$; and $\frac{1}{2} AB = AL = \sin. \frac{1}{2} a = \frac{1}{2} \sqrt{\sin.^2 a + v\text{-sin.}^2 a}$.

205. PROBLEM 3. The sine and cos. of any arc

arc a , being given, to calculate the sine of the double arc $2a$?

SOL. Take the arc $AD = a$ (fig. 50), and suppose its sine, AL and its cos. CL to be known. Produce AL to B ; and the arc AB is double of the given arc AD . Draw the radius CB ; and its perpendicular AP is the sine of $AB = 2AD = 2a$; to find which, take from the two similar triangles CLA , APB : $CA : CL :: AB : AP$; that is, $R : \cos. a :: 2 \sin. a : \sin. 2a = 2 \sin. a \times \cos. a$.

206. PROBLEM 4. The sines and cos. of two arcs a and b being given, to calculate the sine and cos. of their sum $a+b$, and of their difference $a-b$?

SOL. Call the arc AB , a , (fig. 51), and the arc AD , b . Draw the radii CB and CA ; the chord DG perpendicular to CA ; also GR , AS , OL , DP perpendicular to CB ; and EO , GF , perpendicular to DP . Then the sum of the two given arcs $AB+AD=a+b=BD$; and their difference $AB-AD=a-b=AB-AG=BG$. Wherefore, AS is the sine, CS the cos. of A . DO is the sine, CO the cos. of b . DP is the sine, CP the cos. of $a+b$. GR is the sine, CR the cos. of $a-b$. Likewise $DO=GO$. $DE=EF$. $EO=\frac{1}{2}FG=PL=LR$, on account of the similar triangles DEO , DFG : and $OL=EP$. Thence, DP the sine of $a+b$, is equal to $LO+ED$; and GR the sine of the difference $a-b$, is equal to $LO-ED$. CP the cos. of $a+b$, is equal to $CL-EO$; and CR the cos. of $a-b$, is equal to $CL+EO$. Therefore,

1. For

1. For the sine of the sum or of the difference, the values of LO and ED are required, and are to be added or subtracted ; to find which. By the similar triangles CAS, COL; $CA : CO :: AS : OL = \sin. a \times \cos. b$; and by the similar triangles CAS, DOE, (80), $CA : CS :: DO : ED = \sin. b \times \cos. a$. Whence, the sine of the sum $DP = LO + ED = \sin. a + b = \sin. a \times \cos. b + \sin. b \times \cos. a$. And the sine of the difference $GR = LO - ED = \sin. a - b = \sin. a \times \cos. b - \sin. b \times \cos. a$. By a single expression $\sin. a \pm b = \sin. a \times \cos. b \pm \sin. b \times \cos. a$.

2. For the cos. of the sum or of the difference, the values of CL and EO are required, and are to be added or subtracted ; to find which. By the similar triangles CAS, COL, $CA : CO :: CS : CL = \cos. a \times \cos. b$; and by the similar triangles CAS, DEO; $CA : AS : DO : EO = \sin. a \times \sin. b$. Whence $\cos. a \pm b = \cos. a \times \cos. b \pm \sin. a \times \sin. b$.

207. COROL. From the equations (203) which are,

$$\left\{ \begin{array}{l} \text{Cos.} = \sqrt{1 - \sin^2.} \\ \text{Tang.} = \frac{\sin.}{\cos.} \\ \text{Sec.} = \frac{1}{\cos.} \end{array} \right\} \left\{ \begin{array}{l} \text{Cot.} = \frac{1}{\text{tang.}} \\ \text{Cot.} = \frac{\cos.}{\sin.} \\ \text{Co-sec.} = \frac{1}{\sin.} \end{array} \right\} \left. \begin{array}{l} \text{v-sin.} = \\ 1 \pm \cos. \end{array} \right\} \text{May be inferred the following,}$$

$$\left\{ \begin{array}{l} \text{Sin.} = \cos. \times \\ \text{tang.} \end{array} \right\} \left\{ \begin{array}{l} \text{Cos.} = \sin. \times \\ \text{cot.} \end{array} \right\} \left\{ \begin{array}{l} \text{Tang.} = \frac{1}{\text{cot.}} \\ \text{Sin.} = \sqrt{1 - \cos.^2} \end{array} \right\}$$

And

And from the equations (206), which for any two given arcs a and b are,

$$\left\{ \begin{array}{l} \text{Sin. } a+b=\sin. a \times \cos. b + \sin. b \times \cos. a \\ \text{Cos. } a+b=\cos. a \times \cos. b + \sin. a \times \sin. b \end{array} \right\} \text{ May be inferred,}$$

$$\left\{ \begin{array}{l} \text{Sin. } a+b+\sin. a-b=2 \sin. a \times \cos. b \\ \text{Sin. } a+b-\sin. a-b=2 \cos. a \times \sin. b \\ \text{Sin. } a \times \cos. b=\frac{1}{2} \sin. a+b+\frac{1}{2} \sin. a-b \\ \text{Cos. } a \times \sin. b=\frac{1}{2} \sin. a+b-\frac{1}{2} \sin. a-b \end{array} \right\} \text{ Whence}$$

From the 1st

}

$$\left\{ \begin{array}{l} \text{Cos. } a+b+\cos. a-b=2 \cos. a \times \cos. b \\ \text{Cos. } a+b-\cos. a-b=-2 \sin. a \times \sin. b \\ \text{Cos. } a \times \cos. b=\frac{1}{2} \cos. a+b+\frac{1}{2} \cos. a-b \\ -\text{Sin. } a \times \sin. b=\frac{1}{2} \cos. a+b-\frac{1}{2} \cos. a-b \end{array} \right\} \text{ Whence}$$

From the 2nd

}

And from (204, 205) for any given arc a .

Sin.

$$\left\{ \begin{array}{l} \text{Sin. } \frac{1}{2} a = \frac{1}{2} \sqrt{\text{sin.}^2 a + \text{v. sin.}^2 a} \\ \text{Sin. } 2 a = 2 \text{ sin. } a \times \text{cos. } a \end{array} \right\} \begin{array}{l} \text{Cos. } a = \frac{\text{sin. } 2 a}{2 \text{ sin. } a} \\ \text{Sin. } a = \frac{\text{sin. } 2 a}{2 \text{ cos. } a} \end{array}$$

By means of these equations which are either directly demonstrated or inferred from the foregoing problems, the tables of the trigonometrical lines may be easily calculated. For if the sine of an arc of 1° was known, the sines of all the arcs from 1° to 90° could be easily found by the formulæ (205. 206), by taking the sine of 2° ; then the sine of $2^\circ + 1^\circ$, then the sine of 4° , then the sine of $4^\circ + 1^\circ$, then, &c. The same might be done for the sines from $1'$ to $60'$, or from $1''$ to $60''$, if the sine of $1'$ or of $1''$ were known. Then, from the sines all the remaining trigonometrical lines might be calculated by the formulæ (203) or some of the above equations. Therefore,

208. PROBLEM 5. To calculate the sine of an arc of $1'$?

SOL. The sine of an arc of 30° is equal to half the radius or total sine (195.). From this by the formula (204) may be calculated the sines of all the arcs of the geometrical progression $30^\circ : \frac{30^\circ}{2} : \frac{30^\circ}{4} : \frac{30^\circ}{8} : \&c.$

But as an arc of $1'$ is not one of the terms of the progression, take the 12th term, which is an arc of $52'' 44''' 3'''' 45'''''$; and make the proportion; the arc of $52'' 44''' 3'''' 45'''''$ is to the sine of the same, as an arc of $1'$ is to the sine of $1'$. Such a proportion though not mathematically true, will give the sine of $1'$ without a sensible error (202) and may be taken equal to the arc itself of $1' = 0,000290$ (111.)

The same might be calculated by the method of the chords in geometry, taking the sines to be the half chords of the double arcs.

209. Higher branches of mathematics afford more expeditious methods of calculating sines, co-sines, tangents, &c. Our object here is only to shew that the tables of trigonometrical lines may be calculated.

Many such tables have been published. The most recent and correct of all are *Hutton's tables*, London, 1811. They all generally contain the natural sines, cosines, tangents, secants, &c. of all the arcs from $1'$ and sometimes from $1''$ to 45° , which is sufficient for all the arcs from 0° to 90° ; the cosine, co-tangents, &c. from 0° to 45° being the sines, tangents, &c. of all the arcs from 45° to 90° . They likewise contain the Logarithms of their expressions, together with an extensive Table of the Logarithms of the natural numbers, mostly from 1 to 100000; which facilitates calculations, by simplifying multiplications and divisions, powers and roots. They are also very often useful to change the subtraction into an addition, by means of the arithmetical complement, which may be thus briefly explained.

The *arithmetical complement* of a number is the difference between that number and the next decimal number immediately above it. For instance, the arithmetical complement of 54376 is $100000 - 54376 = 45624$, and may be found without writing the decimal number, by subtracting the first figure from 10, and every other figure from 9.

Arithmetical complements are used to change subtractions into additions. To subtract a given number B from another given number A, add to A the arithmetical complement of B, and diminish by a unit the last or highest figure of the sum. For instance, to subtract 54376 from 97689, add to the second number the arithmetical complement of the first $97689 + 45624 = 143313$, and take 1 from the highest figure; the rest $43313 = 97689 - 54376$ the true difference.

The

The reason of it is obvious. Adding the arithmetical complement $45624 = 100000 - 54376$ is adding 100000 and subtracting 54376; whence the result is too great by 100000, which is subtracted by taking one unit from the highest figure.

The arithmetical complements are particularly used in performing divisions by Logarithms. Suppose, for instance, $\frac{432}{48}$. it will be $\text{Log. } 432 - \text{Log. } 48 = \text{Log. } 9$.

The arithmetical complement of $\text{Log. } 48$, which is 1,6812412, is $10,0000000 - 1,6812412 = 8,3187588$, which added to $\text{Log. } 432$, which is 2,6354837, makes a sum 10,9542425; from which taking the last unit, the remainder 0,9542425 is the Logarithm of 9 the quotient of $\frac{432}{48}$.

Resolution of Triangles.

THE resolution of triangles is practical trigonometry. It is particularly used to measure the height or distance of objects which are not conveniently accessible. The right-angled and the oblique-angled triangles have their peculiar method of solution. But the art of measuring angles in the given triangles is previously required. Therefore,

212. PROBLEM. To measure the magnitude of an angle?

SOL. A circle or half circle, or even a quadrant exactly graduated, is the instrument by means of which practical geometry or trigonometry measures the magnitude of angles. One of its diameters is immoveably fixed at the origin of the graduation; and another diameter is moveable about the centre of the circle or

R

quadrant

quadrant. The observator directs the immoveable one to either of two distant objects, and the moveable one to the other. The two visual rays together with an imaginary line joining the two distant objects, make a triangle; and the instrument gives the measure of the angle contained by the two visual rays, the vertex of which is at the centre of the instrument. Such instruments are always provided with a plummet, by means of which the observator may easily get the vertical and horizontal lines. One of these instruments called Graphometer, is exhibited (fig. 52.). The accuracy of the resolution of triangles depends entirely on the observator, and on the accurate graduation of the instrument. It will be sufficient to give the delineation and usage of such instruments in the course of the lectures.

Resolution of Right-angled Triangles.

213. Three things, among which one side, being requisite and sufficient for the resolution of a triangle (191), a right-angled triangle may always be resolved, if, besides the right-angle, any two other things (one being a side) be known. It is to be remarked, that if one of the acute angles be given, the three angles are known, as the two acute ones are complement to each other.

All the possible cases for a right-angle triangle are but four, viz. 1. One of the acute angles with one of the small sides being given. 2. One of the acute angles with the hypotenuse. 3. One of the small sides with the hypotenuse. 4. The two small sides. These four cases may be resolved by the following analogies.

214.

214. ANALOGY 1. In a right-angled triangle, the hypotenuse being taken for radius; the radius of the tables is to the sine of either of the two acute angles, as the hypotenuse is to the side opposite to the same acute angle.

And, the radius of the tables is to the co-sine of either of the two acute angles, as the hypotenuse is to the small side contiguous to the same acute angle.

DEM. As in any triangle whatsoever, the sines of the angles are proportional to the opposite sides (197) the right-angled triangle APC (fig. 48) gives the proportions $\sin. APC : \sin. ACP :: AC : AP$; and $\sin. APC : \sin. PAC :: AC : CP$. Then, as $\sin. APC = R$ the total sine (193. 2); and CP is the co-sine of the contiguous angle ACP, as well as AP the co-sine of CAP (193. 3), the proportions become

$$\left\{ \begin{array}{l} R : \sin. ACP :: AC : AP \\ R : \cos. CAP :: AC : AP \end{array} \right\} \text{ and } \left\{ \begin{array}{l} R : \sin. CAP :: AC : CP \\ R : \cos. ACP :: AC : CP \end{array} \right.$$

215. ANALOGY 2. In a right-angled triangle, one of the small sides being taken for radius; the radius of the tables is to the tangent of either of the acute angles, as the small side contiguous to that acute angle is to the side opposite to the same.

And, the radius of the tables is to the secant of either of the acute angles, as the small side contiguous to that angle is to the hypotenuse.

DEM. If a circle PL be described from the vertex C of one of the acute angles, with the radius CP; then, $CP = R$, AP is the tangent of the acute angle ACP, and AC is its secant. And if the circle was described from the vertex A of the other acute angle, with the radius AP; then, $AP = R$, CP is the tangent, and AC the secant of the acute angle CAP. Whence,

$$\left\{ \begin{array}{l} R : \tan. ACP :: CP : AP \\ R : \sec. ACP :: CP : AC \end{array} \right\} \text{ and } \left\{ \begin{array}{l} R : \tan. CAP :: AP : CP \\ R : \sec. CAP :: AP : AC \end{array} \right.$$

To these analogies may be subjoined the following ones, which, in case of two sides being given together with the right angle, afford more convenient calculations.

$$AC^2 = CP^2 + AP^2 \quad (89). \quad \text{Whence } AC = \sqrt{CP^2 + AP^2}$$

$$\left. \begin{aligned} CP^2 &= AC^2 - AP^2 = \overline{AC + AP} \times \overline{AC - AP} \\ AP^2 &= AC^2 - CP^2 = \overline{AC + CP} \times \overline{AC - CP} \end{aligned} \right\} \quad \text{Whence}$$

$$\left\{ \begin{aligned} \text{Log. } CP &= \frac{1}{2} \text{Log. } \overline{AC + AP} + \frac{1}{2} \text{Log. } \overline{AC - AP} \\ \text{Log. } AP &= \frac{1}{2} \text{Log. } \overline{AC + CP} + \frac{1}{2} \text{Log. } \overline{AC - CP} \end{aligned} \right.$$

These analogies under (214. 215) are sufficient for the resolution of a right-angled triangle in all possible cases ; and the table subjoined exhibits the analogy to be used in every peculiar case.

216. PROBLEM 1. To measure the vertical altitude of the accessible tower AB (fig. 25.)

SOL. At a convenient distance BL from the tower take with the Graphometer the horizontal line OB, as near the foot of the tower as circumstances will allow. Measure the angle AOB, and suppose it to be $50^\circ 30'$. Measure also accurately the horizontal distance OB, and suppose it to be 150 feet. Then, make the proportion $R : \text{tang. } AOB :: OB : AB = \frac{\text{tang. } AOB \times OB}{R}$

$$= \frac{\text{tang. } 50^\circ 30' \times 150}{R}. \quad \text{Which may be thus calculated}$$

1^o by natural sines, 2^o by Logarithms.

$$1^\circ \left\{ \begin{aligned} \text{Tang. } 50^\circ 30' &= 1,2130970 \dots \dots \text{as } R = 1 \\ &\times 150 \end{aligned} \right.$$

$$\left\{ \begin{aligned} \text{Product } AB &= 181,964550 = 181 \text{ feet } 11\frac{1}{2} \text{ inches.} \end{aligned} \right.$$

$$\left\{ \begin{aligned} \text{Log. tang. } 50^\circ 30' &= 10,0838955 \dots \\ + \text{Log. } 150 \text{ feet,} &= 2,1760913 \end{aligned} \right.$$

$$2^\circ \left\{ \begin{aligned} \text{Sum} &= 12,2599868 \\ \text{Log. } R \text{ subtracted} &= 10 \end{aligned} \right.$$

$$\left\{ \begin{aligned} \text{Difference} &= 2,2599868 \text{ Log. } 181,98 \text{ feet} \\ &\text{nearly.} \end{aligned} \right.$$

If

If the station O for the observation were chosen, so as to have OB horizontal and the angle AOB of 45° , the vertical altitude AB should be found, without farther calculation, to be equal to OB. Because, then OB being taken for radius, AB is the tangent of 45° , equal to the radius (195. 2.)

217. PROBLEM 2. To measure the vertical altitude of a mountain, or any other inaccessible height?

SOL. Suppose A (fig. 25.) to be the top of the mountain. At a station O, from which the top A of the mountain is visible; take on the ground a line making a right-angle with a visual ray from O to A, and suppose its length to be 150 feet. Measure also the angle made at the extremity of that line by another visual ray from the top of the mountain A; and suppose that angle to be 75° . Then, the line on the ground being taken for radius, AO is the tangent of the angle of 75° , and its length is found by $R : 150 \text{ feet} :: \text{tang.}$

$$75^\circ : AO = \frac{150 \times \text{tang. } 75^\circ}{R}, \text{ and by Logarithms,}$$

$$\left\{ \begin{array}{l} \text{L. 150 feet} \quad = 2,1760913 \\ + \text{L. tang } 75^\circ \quad = 10,5719475 \\ \hline \text{Sum} = 12,7480388 \\ \text{L. R subtracted} = 10, \\ \hline \text{Difference} = 2,7480388 \dots \text{L. 560 feet} = \\ \text{AO.} \end{array} \right.$$

After thus finding the distance from the station O to the top of the mountain A; let the observator return to O, and take with his Graphometer a horizontal line in the direction OB, and measure the angle AOB; which being supposed to be $50^\circ 30'$, the following proportion will give the vertical altitude of the mountain, which is

is AB. $R : AO :: \sin. AOB : AB$; or $R : 560 \text{ feet} :: \sin. 50^\circ 30' : AB$.

By Logarithms.

$$\left\{ \begin{array}{l} L. 560 \dots\dots\dots = 2,7481880 \\ + L. \sin. 50^\circ 30' = 9,8874061 \\ \hline \text{Sum} = 12,6355941 \\ L. R \text{ subtracted} = 10, \\ \hline \text{Difference} = 2,6355941 \dots L. 432 \text{ feet} = AB. \end{array} \right.$$

the vertical altitude of the top of the mountain above the horizontal level of the place of observation O. (the fractions of a foot have been neglected in the above calculations.)

Resolution of Oblique-angled Triangles.

218. Oblique-angled triangles are those which have no right angle. Their resolution comprehends four different cases. 1° Two angles and one side being given. 2° Two sides being given, with one of their opposite angles. 3° Two sides being given, with the angle which they contain. 4° The three sides being given.

219. 1st Case. Two angles and one side being given, (the third angle is known, being the supplement of the two given angles.) It may be resolved immediately by the proportionality of the sides to the sines of their opposite angles (197) by the following proportions :

$$\text{Being given, (fig. 53)} \left\{ \begin{array}{l} B, D, AB \dots\dots \sin. D : AB :: \sin. B : AD :: \sin. A : BD \\ B, D, AD \dots\dots \sin. B : AD :: \sin. D : AB :: \sin. A : BD \\ B, D, BD \dots\dots \sin. A : BD :: \sin. B : AD :: \sin. D : AB \end{array} \right.$$

2nd

2nd Case. Two sides being given, with one of their opposite angles. This case may also be resolved immediately by the proportionality of the sides to the sines of their opposite angles, by the following proportions :

Being $\left\{ \begin{array}{l} AB, AD, D \dots AB : \sin. D :: AD : \sin. B \\ AB, AD, B \dots AD : \sin. B :: AB : \sin. D \end{array} \right.$

Then $\sin. B : AD :: \sin. A : BD$

As the trigonometrical calculations suppose the triangle to be resolved similar to another triangle, the dimensions of which are the dimensions of the tables : This 2nd case must be a dubious case in trigonometry as it is in geometry (82.) AP being supposed to be perpendicular on BD and AD=AL, the angles D and ALB are supplement to each other, and have the same sine. Therefore, unless it is previously known whether the angle D is acute or obtuse, the side opposite to the angle A may be equally BD or BL, and the angle opposite may be BAD or BAL.

220. 3rd Case. Two sides being given, with the angle which they contain. The angle given in this case not being opposite to one of the given sides, the proportionality of the sines and sides cannot be applied immediately. It requires the intermediate application of the analogy (201), which in the triangle BAD (fig. 53) is (being given AB, AD, A) $AB+AD : AB-AD :: \text{tang. } \frac{D+B}{2} : \text{tang. } \frac{D-B}{2}$. As the sum of the two angles B+D is the supplement of the given angle A, the three first terms of the proportion are known ; and the 4th term $\text{tang. } \frac{D-B}{2}$ gives the difference of the two angles which are not given. The sum and the difference

ference of D and B being known, the greater of the two

$$D = \frac{D+B}{2} + \frac{D-B}{2} \text{ and the lesser } B = \frac{D+B}{2} - \frac{D-B}{2}.$$

Now, the three angles of the triangle ABD being known, the third side BD is found by the proportion

$$\sin. B : AD :: \sin. A : BD = \frac{\sin. A \times AD}{\sin. B}.$$

221. 4th Case. The three sides being given. There are two principal solutions of this case.

SOL. 1. This requires the intermediate application of the analogy (198), (being given BD, AB, AD of the triangle ABD, and BL, AB, AD=AL in the triangle ALB). A perpendicular AP being drawn from the angle A to its opposite side BD or BL, the analogy is BD or BL : AB+AD :: AB-AD : BP+DP ; that is,

$$\text{For the } \begin{cases} \text{triangle } \{ \text{BAD. } BD = BP + DP : AB + AD :: AB - AD : BP - DP \\ \text{BAL. } BL = BP - DP : AB + AD :: AB - AD : BP + DP \end{cases}$$

So, if the triangle to be resolved is BAD, the sum of the two segments $BD = BP + DP$ is given, and the analogy gives their difference $BP - DP$. And if the triangle to be resolved is BAL, the difference of the two segments $BP - DP = BL$ is given, and the analogy gives their sum $BP + DP$. Whence, the two segments

$$BP \text{ and } DP \text{ are known, viz. } BP = \frac{BP + DP}{2} + \frac{BP - DP}{2};$$

$$\text{and } DP = \frac{BP + DP}{2} - \frac{BP - DP}{2}.$$

Now, the given triangle BAD or BAL is divided by the perpendicular AP into two right-angled triangles BAP and DAP or LAP ; and (the angle A being given) the other two angles B and D or L, are calculated by the analogies $AB : BP :: R : \cos. B$; and $AD : DP :: R : \cos. D$, or $AL = AD : LP :: R : \cos. ALP = \cos. ALB$.

SOL 2. *Analogy.* To find the angle ADB (Fig. 54) the three sides of the triangle BAD being given

$$AD \times DB : \frac{AD+DB+AB}{2} - AD \times \frac{AD+DB+AB}{2} - BD ::$$

$$R^2 : \sin^2 \frac{1}{2} ADB$$

DEM. Suppose AP perpendicular to BD. Describe the arc AO from the centre D. Draw the chord AO, and DL perpendicular on AO; so that AD=DO, AL=OL, ADL=BDL= $\frac{1}{2}$ ADB.

$$\text{Let } \begin{cases} AD=a \\ AB=b \\ BD=c \end{cases} \begin{cases} AP=y \\ DP=x \\ BD-DP=c-x \end{cases} \begin{cases} PO=DO-DP=AD \\ -DP=a-x \\ a+b+c=2s. \\ \frac{a+b+c}{2}=s. \end{cases}$$

By the right angled triangle APD, $x^2 + y^2 = a^2$

By the right angled triangle APB, $c^2 - 2cx + x^2 + y^2 = b^2$

If the second equation be subtracted from the first; $2cx - c^2$

$$= a^2 - b^2; \text{ and } x = \frac{a^2 - b^2 + c^2}{2c}$$

By the right angled triangle APO, $AO^2 = y^2 + a^2 - 2ax + x^2$. Substituting in the last a^2 instead of $x^2 + y^2$

and $\frac{a^2 - b^2 + c^2}{2c}$ instead of x , and $2s$ instead $a + b + c$; comes

$$AO^2 = y^2 + a^2 - 2ax + x^2 = 2a^2 - 2ax = 2a \times \overline{a-x} = 2a \times a + \frac{b^2 - a^2 - c^2}{2c} = 2a \times \frac{2ac + b^2 - a^2 - c^2}{2c} = \frac{2a}{2c} \times \overline{b^2 - a^2} +$$

$$\frac{2ac - c^2}{c} = \frac{a}{c} \times \overline{b^2 - a^2} + 2ac - c^2 = \frac{a}{c} \times \overline{b+c-a} \times \overline{b-c+a}$$

$$= \frac{a}{c} \times \overline{a+b+c-2a} \times \overline{a+b+c-2c} = \frac{a}{c} \times \overline{2s-2a} \times \overline{2s-2c}$$

$$= \frac{4a}{c} \times \overline{s-a} \times \overline{s-c}$$

By the right angled triangle DLO, $DO : OL :: R : \sin ODL$; that is, $a : OL = \frac{1}{2} AO :: R : \sin. ODL = \sin. \frac{1}{2} ADO$

S

= Sin.

$$= \text{Sin. } \frac{1}{2} \text{ ADB}; \text{ and } \frac{1}{2} \text{ AO} = \frac{a \times \text{Sin. } \frac{1}{2} \text{ ADB}}{R}; \text{ or } \text{AO} =$$

$$\frac{2a \times \text{Sin. } \frac{1}{2} \text{ ADB}}{R}; \text{ Whence } \text{AO}^2 = \frac{4a^2 \times \overline{\text{Sin. } \frac{1}{2} \text{ ADB}}^2}{R^2}; \text{ and}$$

taking into one equation the two expressions of AO^2 , comes

$$\frac{4a^2 \times \overline{\text{Sin. } \frac{1}{2} \text{ ADB}}^2}{R^2} = \frac{4a}{c} \times \overline{s-a} \times \overline{s-c}. \text{ Banishing the frac-}$$

tions;

$$4a^2 c \times \overline{\text{Sin. } \frac{1}{2} \text{ ADB}}^2 = 4a \times R^2 \times \overline{s-a} \times \overline{s-c}. \text{ Dividing by } 4a;$$

$$ac \times \overline{\text{Sin. } \frac{1}{2} \text{ ADB}}^2 = R^2 \times \overline{s-a} \times \overline{s-c}. \text{ Whence the proportion}$$

$$ac : \overline{s-a} \times \overline{s-c} :: R^2 : \overline{\text{Sin. } \frac{1}{2} \text{ ADB}}^2. \text{ And substituting again the sides of the triangle BAD}$$

$$\text{AD} \times \text{BD} : \frac{\overline{\text{AD} + \text{DB} + \text{AB}}}{2} - \text{AD} \times \frac{\overline{\text{AD} + \text{DB} + \text{AB}}}{2} - \text{BD} ::$$

$$R^2 : \overline{\text{Sin. } \frac{1}{2} \text{ ADB}}^2$$

This second Solution, though apparently complicate, is simpler and more expeditious than the first in the trigonometrical calculations by logarithms. Half the angle ADB is found by this equation.

$$2 \text{ Log. Sin. } \frac{1}{2} \text{ ADB} = \text{Log. } \frac{\overline{\text{AD} + \text{AB} + \text{BD}}}{2} - \text{AD} +$$

$$\text{Log. } \frac{\overline{\text{AD} + \text{AB} + \text{BD}}}{2} - \text{BD} + 2 \text{ L. R.} - \text{Log. AD} - \text{Log. BD}$$

Here follow examples of the five foregoing Solutions,

223. PROBLEM of the 1st Case. Suppose the triangle ABD (Fig. 53.) to be resolved, in order to calculate the distances from the inaccessible place A to the accessible places B and D.

SOL. Measure the angle B supposed $22^\circ 40'$, the angle D supposed $75^\circ 15'$, which leave the angle A of $82^\circ 5'$, and the side BD supposed to be 2765 yards. Then (219)

Sin.

$$\text{Sin } A : 2765 :: \left\{ \begin{array}{l} \text{Sin. B} : AD = \frac{\text{Sin. B} \times 2765}{\text{Sin. A}} \\ \text{Sin. D} : AB = \frac{\text{Sin. D} \times 2765}{\text{Sin. A}} \end{array} \right\} \text{By Log.}$$

$$\text{Log. AD} = \left\{ \begin{array}{l} 9,5858771 \dots \text{Log. Sin. B} = 22^\circ 40' \\ + 3,4416951 \dots \text{Log. 2765} \\ + 0,0041589 \dots \text{Arith. Compl. of } 9,9958411 \\ \text{Log. Sin. A} = 82^\circ 5' \\ = 3,0317311 \dots \text{Log. 1075, 8 yards} = AD \end{array} \right.$$

$$\text{Log. AB} = \left\{ \begin{array}{l} 9,9854471 \dots \text{Log. Sin. D} = 75^\circ 15' \\ + 3,4416951 \dots \text{Log. 2765} \\ + 0,0041589 \dots \text{Arith. Compl. of Sin. A} = 82^\circ 5' \\ = 3,4313011 \dots \text{Log. 2699, 6 yards} = AB \end{array} \right.$$

224. The same solution may be used to measure the vertical attitude AP (Fig. 55.) of a flying object A, such as a cloud, an areostad, a meteor, &c.; viz. two observers, on the same horizontal level, one at D, the other at B, measure at the same moment, the two angles ADP, ABP made by the visual lines AD and AB with the horizontal DP. Then, after measuring DB, they may resolve the triangle ADB, in which the obtuse angle ABD is the supplement of ABP, and find AD or AB as (223). Remains only to resolve the right angled triangle APD or APB by the analogy (214) R : AD or AB :: Sin. ADP or Sin ABP : AP.

The same solution might be used to measure the vertical altitude of a mountain (217). From any two stations B and D measure the two angles B and D of the triangle BAD, in order to calculate either of the two sides AB or AD. Then taking from D or B a horizontal line DP or BP, resolve the right angled triangle APD or APB, to find AP the vertical altitude of the mountain.

The same might be done from two stations at the top of the mountain, by taking there the base of a triangle, of which AD should be a side, and forming with AD and the vertical line

line AP a right angled triangle, in which AP should be calculated

225. PROBLEM of the 2d Case. In the triangle ABD (Fig. 53). Being given the side AB=465 yards, the side AD=288 yards, and the angle B=36° 30' opposite to AD; the angle D is found by the proportion AD:Sin B::AB:Sin

$$D = \frac{\sin B \times AB}{AD} \quad \text{By Logarithms}$$

$$\begin{array}{r} \text{Log. Sin B} = 36^\circ 33' = 9,7743876 \\ \text{Log. AB} = 465 \quad + 2,6674530 \end{array} \left. \vphantom{\begin{array}{r} \text{Log. Sin B} \\ \text{Log. AB} \end{array}} \right\} \text{Sum} = 12,4418406$$

$$\text{Log. AD} = 288 \dots \text{subtracted} \dots \dots \dots - 2,4593925$$

$$\text{Difference} \left\{ \begin{array}{l} \text{Log. Sin D acute} = 79^\circ 21' \\ \text{Log. Sin D obtuse } 100^\circ 39' \end{array} \right\} = 9,9924481$$

Whence the angle A = $\begin{cases} 64^\circ 9' & \text{when D is acute} \\ 42^\circ 51' & \text{when D is obtuse} \end{cases}$

the two sides BD or BL may be calculated as (223)

226. PROBLEM of the 3d Case. Let it be asked what the distance is from the place B to the place D (Fig. 53.) both accessible from the place A, not accessible from one to the other?

SOL. Measure the angle BAD supposed to be 48°. Suppose the side AB=142 furlongs, and the side AD=120 fur-

longs. By (220) $AB+AD:AB-AD::\text{tang.} \frac{D+B}{2} : \text{tang.}$

$\frac{D-B}{2}$; that is $142+120=262:142-120=22::\text{tang.}$

$$\frac{180^\circ - 48^\circ}{2} = 66^\circ : \text{tang.} \frac{D-B}{2} = \frac{\text{tang. } 66^\circ \times 22}{262}$$

By Logarithms

$$\left\{ \begin{array}{l} 10,3514169 \dots \text{Log, tang. } 66^\circ \\ + 1,3424227 \dots \text{Log. } 22 \\ + 7,5816987 \dots \text{Arith compt. of } 2,4183013 \text{ Log. } 262 \\ = 9,2755383 \dots \text{Log. tang. } 10^\circ 41' = \text{tang.} \frac{D-B}{2} \end{array} \right.$$

Whence

Whence $\left\{ \begin{array}{l} \text{The angle ADB} = 66^\circ + 10^\circ 41' = 76^\circ 41' \\ \text{The angle ABD} = 66^\circ - 10^\circ 41' = 55^\circ 19' \end{array} \right\} (220)$

Now the side BD is calculated by the proportion (219)

$$\sin 76^\circ 41' : 142 :: \sin 48^\circ : BD = \frac{\sin 48^\circ \times 142}{\sin 76^\circ 41'}$$

By logarithms.

$$\left\{ \begin{array}{l} 9, 8710735 \dots \text{Log. } 48^\circ \\ + 2, 1522883 \dots \text{Log. } 142 \\ + 0, 0118372 \dots \text{Arith. compt. of } 9,9881628 \text{ Log. } \sin 76^\circ 41' \\ = 2, 0351990 \dots \text{Log. } 108 \text{ furlongs, nearly.} \end{array} \right.$$

But, it may happen that the two places. B and D are not accessible from A. If so, two bases must be taken from A, in order to form two triangles, in one of which AB will be a side, in the other AD will be a side, by means of these AB and AD may be calculated as (223).

227. PROBLEM of the 4th Case. Being given, the three sides of the triangle ABD (Fig. 53); viz. BD = 184, AB = 142, AD = 64; to calculate the three angles.

SOL. I. With a perpendicular AP from the angle A to the opposite side (221) $BD : BA + AD :: BA - AD : BP - DP.$ $= \frac{206 \times 78}{84} = 87,326.$ Whence the small segment

$$DP = \frac{184 - 87,326}{2} = 48,337.$$

Now, knowing in the right angled triangle APD the two sides AD, DP, the angle PAD is calculated by the proportion

$$AD : DP :: R : \sin PAD = \frac{R \times 48,337}{64} \text{ and by Logar-}$$

ithms,

$$\left\{ \begin{array}{l} 1, 6842589 \dots \text{Log. } 48,337 \\ + 10, \dots \dots \dots \text{Log. } R \\ + 8, 1938200 \dots \text{Arith. compt. of } 1,8061300 \text{ Log. } 64 \\ = 9, 8780789 \dots \text{Log. } 49^\circ 2' = PAD \\ \text{Therefore } 40^\circ 58' = ABD \end{array} \right\} \text{Sum } 90^\circ$$

Having

Having the angle ADB and the three sides, the following proportion gives the angle ABD. $AB : \sin D :: AD : \sin B$;

$$\text{or } 142 : \sin. 40^\circ 58' :: 64 : \sin. ABD = \frac{\sin. 40^\circ 58' \times 64}{142}$$

By Logarithms.

$$\left\{ \begin{array}{l} 9, 8166521 \dots \text{Log. Sin } 40^\circ 58' \\ + 1, 8061800 \dots \text{Log. 64} \\ + 7, 8477117 \dots \text{Arith compt. of } 2.1522883 \text{ Log. 142} \\ = 9, 4705438 \dots \text{Log. } 17^\circ 12' = ABD \end{array} \right.$$

Wherefore the three angles of the triangle ABD are found, the angle D $= 40^\circ 58'$, the angle B $= 17^\circ 12'$, the angle A $= 121^\circ 50'$.

SOL. II. By (222) $2 \text{ Log. Sin } \frac{1}{2} ADB = \text{Log.}$

$$\frac{AD + BD + AB}{2} - AD + \text{Log.} \frac{AD + BD + AB}{2} - BD$$

$$+ 2 \text{ Log. R} - \text{Log. AD} - \text{Log. BD.}$$

Which, keeping the same values of the three sides as in the last solution.

$$\left\{ \begin{array}{l} 2, 1172713 \text{ Log. } \left\{ \begin{array}{l} AD = 64 \\ + AB = 142 \\ + BD = 184 \end{array} \right\} = \frac{390}{2} = 195 - 64 = 131 \\ + 1, 0413927 \dots \text{Log.} \dots \dots \dots \frac{390}{2} = 195 - 184 = 11 \\ + 20, \dots \dots \dots 2 \text{ Log. R} \\ + 15, 9290022 \text{ sum of Arith. comp. of } \left\{ \begin{array}{l} \text{Log. 64} \\ \text{Log. 184.} \end{array} \right. \\ = 39, 0876662. \end{array} \right.$$

From which taking two units of the last figure on account of the two arithmetical complements and dividing by 2, comes.

$$\frac{19, 0876662}{2} = 9, 5438331. \text{ Log. Sin } 20^\circ 29' = \frac{1}{2} ADB.$$

Whence the angle ADB $= 40^\circ 58'$, the same as by (Sol. 1.)

228. PROBLEM 5. To draw trigonometrically the plan of an extensive country?

SOL.

SOL. Two elevated stations must be selected, from both of which all the principal objects in the proposed country are visible. The distance between these two stations being measured is made the base of as many triangles as there are principal objects, and the two angles of every triangle at the two extremities of that common base are measured. Then every triangle is resolved by the analogies (219) and the distance of every principal object to either of the two stations or to both is calculated. The respective angles give also their respective situations. This shall be exemplified in the explanation.

A TABLE OF THE ANALOGIES

For the resolution of all the cases of right angled *plane triangles*. A being the right angle, B and C the two acute angles, BC the hypotenuse, AB opposite to C, AC opposite to B.

Being given		To find	ANALOGIES.
1	AB, AC	BC	$BC = \sqrt{AC^2 + AB^2}$ or $AB : AC :: R : \text{tang. B}$, Since $\text{Sin. B} : R :: AC : BC$
2		B	$AB : AC :: R : \text{tang. B}$.
3		C	$AC : AB :: R : \text{tang. C}$.
4	AB, BC	AC	$AC = \sqrt{BC^2 - AB^2}$. or $\text{Log. AC} = \frac{1}{2} \text{Log. (BC + AB)} + \frac{1}{2} \text{Log. (BC - AB)}$.
5		B	$BC : AB :: R : \text{Cos. B}$.
6		C	$BC : AB :: R : \text{Sin. C}$.
7	AC, BC	AB	$AB = \sqrt{BC^2 - AC^2}$ or $\text{Log. AB} = \frac{1}{2} \text{Log. (BC + AC)} + \frac{1}{2} \text{Log. (BC - AC)}$.
8		B	$BC : AC :: R : \text{Sin. B}$.
9		C	$BC : AC :: R : \text{Cos. C}$.
10	AB, B	AC	$R : \text{tang. B} :: AB : AC$.
11		BC	$\text{Cos. B} : R :: AB : BC$.
12	AB, C	AC	$R : \text{cot. C} :: AB : AC$.
13		BC	$\text{Sin. C} : R :: AB : BC$.
14	AC, B	AB	$R : \text{cot. B} :: AC : AB$.
15		BC	$\text{Sin. B} : R :: AC : BC$.
16	AC, C	AB	$R : \text{tang. C} :: AC : AB$.
17		BC	$\text{Cos. C} : R :: AC : BC$.
18	BC, B	AB	$R : \text{cos. B} :: BC : AB$.
19		AC	$R : \text{Sin. B} :: BC : AC$.
20	BC, C	AB	$R : \text{Sin. C} :: BC : AB$.
21		AC	$R : \text{cos. C} :: BC : AC$.

SPHERICAL TRIGONOMETRY.

230. Spherical Trigonometry is the science and art of solving this general problem; viz. out of the six parts, three angles and three sides of a spherical triangle, any three being given to find out the other three. As in plane, so in spherical Trigonometry two data are insufficient for the resolution of a triangle; but in this, the three sides may be calculated from the three angles. The parts of a spherical triangle are calculated by means of the sines, tangents and secants of its angles and of its sides, the theory of which was already given in plane trigonometry.

A *Spherical Triangle* is a triangle made on the surface of a sphere by the meeting of three arcs of its great circles.

Therefore spherical trigonometry has to give the general properties of circles, which are plane sections of the sphere, of the spherical angles made at the meeting and intersection of their circumferences, and of spherical triangles; the peculiar properties of right angled and oblique angled spherical triangles, and the analogies by which they are resolved; with a few practical examples of the principal cases.

Properties of Circles.

231. Definitions. A sphere is a solid circular in every direction; and may be considered as generated
T
by

by the revolution of a circle about an immoveable diameter (166); so that every point of its surface is equidistant from its centre, all its radii are equal, and all its plane sections are circles.

A great circle is the circumference of a plane section through the centre of the sphere. Whence, the centre of the sphere is the common centre of all the great circles, their planes bisect the sphere, the radius of the sphere is their common radius, and they are all equal.

A small circle is the circumference of a plane section which does not pass through the centre of the sphere. Small circles are not used in spherical trigonometry, on account of their not wanting variable centre and magnitude; their arcs, when they occur, are calculated by equal arcs of great circles.

A straight line through the centre, and perpendicular to the plane of a great circle, is the *axis* of that circle; and its two extremities are the *Poles* of the same.—Hence, two great circles cannot have a common axis or common poles; and each of the poles is 90° distant from every point of the circumference, since the axis makes right angles with all the diameters of the circular plane section (138).

232. COROLL. 1. The spherical distance of two points on the surface of the sphere, is measured by an arc of a great circle joining the two points: and the distance from a point to a circle is measured by an arc of a great circle perpendicular to it. Because such arcs are constant and invariable; whereas arcs of
small

small or oblique great circles are of a variable magnitude.

2. Two perpendicular great circles pass by the two poles of each other. Because the planes of two perpendicular great circles intersecting each other, the axis of each is in the plane of the other. The same is to be said of two perpendicular arcs of great circles, if sufficiently produced.

3. The distance between the two intersections of two great circles is constantly 180° . Because the intersection of their two planes is a straight line which passes through the centre of the sphere, and is a diameter of each circle (148). The same cannot be said of small circles.

4. When a great circle is perpendicular to two other great circles, its two poles are at the two intersections of the other two; and reciprocally.

5. When a sphere resolves about the axis of its great circle, every point of its surface describes a small circle, the plane of which is parallel to the plane of the great one, and the centre of every one is in that axis of rotation. Therefore, the radii of such small circles are the sines of their distances from the next pole, or the cosines of their distances from the circumference of the parallel great circle; whence, the magnitudes of such parallel circles are proportional to these sines or cosines (98).

6. Two great circles perpendicular to a third great circle, divide it and all its parallel small circles into proportional parts, or into arcs of the same number of degrees. Whence, two great circles are at the

greatest distance from each other, at 90° from their intersection.

Properties of Spherical Angles.

233. A *Spherical Angle* is the divergency of two arcs of great circles. Consequently, a spherical, like a rectilineal angle may increase from 0° to 180° , where the two arcs coincide with the same circle.

But, the divergency of the corresponding parts of the two sides of a spherical angle is not constant through all their length, like the divergency of the corresponding parts of the two sides of a rectilineal angle. It decreases gradually from the vertex to the distance of 90° , where it vanishes, the two corresponding infinitely small arcs being parallel, since they are perpendicular to the same great circle, the pole of which is at the vertex of the angle: and, from thence, their corresponding parts converge again, and form another angle equal to the first at the distance of 180° from the vertex of the first.

Therefore a spherical angle is taken to be the same as the angle made by the two tangents of its sides at the vertex; which tangents are in the planes of the two great circles to which the two sides belong, and perpendicular to the line of their intersection. Whence,

234. THEOREM 1. A spherical angle made by two arcs of great circles is the same as the angle made by the planes of the two great circles.

DEM. The spherical angle of two arcs of great circles

circles is the same as the angle of their two tangents at the vertex. These tangents are in the planes of the two great circles, and perpendicular to the line of their intersection. Of course, the angle of the two tangents is the same as the angle of the two planes of the two sides (151).

235. THEOREM 2. The measure of a spherical angle is the arc of a great circle (or of any of its parallel small circles) contained by its two sides, and having its pole at the vertex.

DEM. The spherical angle BAD (Fig. 56.) is by the last, the same as the angle of the two planes of the two great circles ABG , ADG intersecting each other in the line AG : and the angle of the two planes is equal to the rectilineal angle BCD of the two radii of the sphere BC , DC in the two planes and perpendicular to AG (151). But, the measure of this is the arc BD of a great circle $RBDL$, having its two poles at A and G , since it is central to it.

If the two lines FO , EO be in the planes of the two sides of the angle BAD , and both perpendicular on the same point O of the line of intersection AG , the arc FE is an arc of a small circle parallel to and of the same number of degrees as the arc BD (232, 5). Then the arc FE may be taken as well as BD for the measure of the spherical angle BAD .

The same may be demonstrated for the spherical angle BGD made by the prolongation of the sides of the angle BAD at their second intersection G .

236. COROLL. 1. Spherical angles may be
right,

right, acute or obtuse, according as they are measured by an arc equal to, less or greater than 90° .

2. When two great circles intersect each other, they make four angles, the sum of which is 360° or four right angles; any two contiguous angles are supplementary to each other, their sum being 180° : and their vertical angles are equal. The same as in lines or planes.

3. A great circle, which has its two poles at the common intersection of two or more great circles, makes right angles with them all; and reciprocally.

4. The arc of a great circle joining the two poles of two great circles is equal to, and may be taken for the measure of the spherical angle at the intersection of the two circles. Because, the two axes being perpendicular to the planes of the circles, make at the centre of the sphere an angle equal to the angle of two radii drawn in the planes perpendicular to the line of their intersection (42. 5), which is the angle of the two planes.

5. If a great circle revolves about one of its diameters which coincides with the axis of another circle, it makes right angles with it constantly. But, if it revolves about one of its diameters, which does not coincide with the axis of the other, the angle at the intersection of the two circles varies continually.

General Properties of Spherical Triangles.

237. A *spherical triangle* is a triangle formed on the surface of a sphere, by the meeting of three arcs of its great circles. Whence,

1°.

1°. Any side of a spherical triangle is always an arc less than a semicircle or 180° . For if the semicircle ABG (Fig. 56.) could be a side of a spherical triangle, one of the other two sides originating at A should, if produced, intersect it again at G; and the other originating at G should, if produced, intersect it again at A (232. 3). Therefore, these two sides cannot meet, unless they coincide with the same semicircle like ADG; and they can form no triangle with ABG.

2°. The sum of any two sides of a spherical triangle is always greater than the third side. This is as obviously true of three circular lines of the same radius, as of the three sides of a rectilineal triangle. Besides, supposing BC, DC, GC to be three radii of the sphere, the spherical triangle BDG consists of three plane angles BCG, BCD, DCG, each having its vertex at the centre C of the sphere. The sum of the two plane angles $BCD + DCG > BCG$, since the point D cannot be on the arc BG. Therefore the measure of the two plane central angles $BCD + DCG$, which is $BD + DG > BG$ the measure of BCG.

3. The sum of the three sides of a spherical triangle is always less than 360° . For, if $BG + BD + DG$ was equal to 360° , the three plane angles $BCG + BCD + DCG$ should also be equal to 360° , and, of course, they should form the entire plane of a great circle, and the three sides, which are the measure of the three plane angles, be the same great circle, and make no angles with one another.

238. THEOREM 1. If from the three angles of
a spherical

a spherical triangle ABD (Fig. 57), as poles, three arcs be described, viz. EG from A, FG from B, EF from D, these three arcs form another spherical triangle EGF; and the two triangles are *supplementary* to each other; that is, every angle of one triangle is supplement to the opposite side of the other.

DEM. By supposition, the angle A is the pole of the arc EG. Therefore, the distance from the vertex A to every point of the arc EG is 90° (231), and the arcs AR and AP are each of 90° . For the same reason, $BL = 90^\circ$, as B is the pole of FG; and $DK = 90^\circ$, as D is the pole of EF.

Likewise, B and D are 90° distant from F; and F is the pole of BD; and so is E the pole of AD, and G the pole of AB. Therefore, $ER = 90^\circ$, and $GP = 90^\circ$. Whence, $ER + GP = EG + PR = 180^\circ$. But PR is the measure of the angle A (235). Of course, the angle A of the triangle ABD and the side EG opposite to A in the triangle EFG are supplementary to each other. A similar demonstration will shew that D and EF, B and FG are supplementary to each other. Then, the angles of the triangle ABD are the supplements of the opposite sides of the triangle EFG.

Likewise, G is the pole of the arc AB, and PL the measure of the angle G. But $AP = 90^\circ$ and $BL = 90^\circ$. Then, $AP + BL = PL + AB = 180^\circ$; and the angle G is the supplement of the side AB. In like manner, F is found to be the supplement of BD, and E the supplement of AD.

Therefore ABD and EGF are supplementary triangles. The demonstration being applicable to the two figures, ascertains the theorem to be true of any triangle

triangle whatsoever, except the triangle of which the three angles and the three sides are of 90° each. But such a triangle has nothing left to be calculated.

239, COROLL. 1. The sum of the three sides of a spherical triangle, together with the sum of the three angles of its supplementary triangle is $3 \times 180^\circ = 540^\circ = 6$ right angles.

2. The sum of the three angles of a spherical triangle is always greater than 180° , and less than 540° .

DEM. The sum of the three angles of a triangle together with the sum of the three sides of its supplementary triangle is a constant sum of 6 right angles, by the last; and the sum of the three sides is always less than four right angles (237, 3). Therefore the sum of the three angles is the difference between 6 right angles and the sum of the three sides; which difference is greater than 2 right angles and less than 6 right angles. Whence,

3. The sum of the three angles of a spherical triangle is not constant, like the sum of the three angles of a rectilineal triangle; but may vary from 180° to 540° ; and of course, the three angles of a spherical triangle may be all acute, all right, all obtuse: and the third angle cannot be inferred from the other two in spherical as in plane trigonometry.

240. THEOREM 2. Two spherical triangles are equal, when they have each: 1 $^\circ$ an equal side, between two equal angles: 2 $^\circ$ an equal angle contained by equal sides: 3 $^\circ$ their three sides equal each to each: 4 $^\circ$ their three angles equal to each.

DEM. 1 $^\circ$. If two spherical triangles have each, an
 U equal

equal side between two equal angles, their two equal sides superimposed must coincide; and so will the other two sides coincide in order to make equal angles with the third.

2°. If two spherical triangles have each, an equal angle contained by equal sides; the two equal angles if superimposed coincide; and so do their two equal sides containing the equal angles.

3°. If two spherical triangles have their three sides equal to each. Superimpose one of their equal homologous sides. Then, the other two homologous sides will also coincide. Because, on the surface of a sphere, where the distances are measured by the arcs of equal circles (232), as well as on a plane surface, where the distances are measured by straight lines, there is but one point which is at given distances from two given points.

4°. If two spherical triangles have their three angles equal each to each. Suppose a triangle abd the three angles of which are equal to the three angles of the triangle ABD each to each (Fig. 57), $A=a$, $B=b$, $D=d$. The three sides of the supplementary triangle of the triangle abd , supposed to be egf , are equal to the three sides of the triangle EGF , $EG=eg$, $GF=gf$, $FE=fe$ (238). Therefore, by the last, the two triangles $egf=EGF$, and $E=e$, $G=g$, $F=f$; but e is supplement to AD as well as to ad , g is supplement to AB as well as to ab , f is supplement to BD as well as to bd . Therefore the three sides of abd and of ABD have each equal supplements, and are equal each

each to each. Then, the three angles being equal, the three sides are likewise equal.

241. THEOREM 3. In an Isosceles spherical triangle the two angles opposite to the two equal sides are equal, and reciprocally.

DEM. In the triangle ABD (Fig. 58) suppose the two sides, $AB=AD$; then the two opposite angles ABD and ADB will be equal. For, take on the equal sides $AG=AL$, which leaves $BG=DL$, and draw the two arcs BD and DG. The two triangles ALB, AGD are equal, on account of a common angle A contained by equal sides (240). Therefore the third sides $BL=DG$. Whence the two triangles GBD, LDB are equal, on account of their three sides equal each to each (240). Of course, the homologous angles GBD, LDB opposite to the two equal sides AB, AD are equal.

Reciprocally, if the two angles $\angle ABD$, $\angle ADB$ are equal, their opposite sides AD , AB will be equal. For, take $BG = DL$, and draw BL and DG . Then, the two triangles $\triangle BDL = \triangle DBG$, on account of the two equal angles B and D contained by equal sides. Then the two homologous angles $\angle BGD = \angle BLD$, and also their supplements $\angle AGD = \angle ALB$ (236. 2). Likewise the two homologous angles $\angle BDG = \angle DBL$; and $\angle ABD - \angle DBL = \angle ABL = \angle ADB - \angle BDG = \angle ADG$. Therefore, the two triangles $\triangle AGD = \triangle ALB$, on account of their three angles being equal each to each (240); and their two homologous sides $AG = AL$. Of course, $AG + GB = AB = AL + LD = AD$.

242. COROLL. 1. An Equilateral triangle is
U 2 equiangular,

equiangular, and reciprocally. It follows immediately from the theorem. But it is to be remarked, that the magnitude of the angles is not determined (239.3). They are known only to be greater than 60° each (239.2): except when the three sides are 90° each in which case the three angles are right angles (235) being measured by the opposite sides. And when the three sides are greater than 90° , the three angles are obtuse.

2. In a spherical triangle the greatest side is opposite to the greatest angle, and the smallest side to the smallest angle; and reciprocally.

DEM. Let the triangle AFB (Fig. 59) have its three sides unequal, BF the greatest. Take a point D equidistant from A and B, and draw AD. Then $AD=BD$ (232), and the angles $BAD=ABD$ (241). But the side $BF=BD+DF=AD+DF>AF$ (237.2); and the angle BAF opposite to BF is greater than the angle B opposite to AF. Since $BAD=ABD$.

The demonstration of the reciprocal is the same.

PROPERTIES

Of Right Angled Spherical Triangles.

243. In a right-angled spherical triangle the side opposite to the right angle, or to the angle considered as a right angle, is called *Hypothénuse*; and the other two angles or sides of whatever magnitude are called *Oblique* angles and *Oblique* sides.

244. THEOR. 1. Each of the two oblique angles
of

of a right angled spherical triangle is of the same kind as its opposite side ; that is, less, equal to or greater than 90° as its opposite side is less, equal to or greater than 90° .

DEM. Let the angle A (Fig. 60) be a right angle, produce AB to C and E, so that $AC = 90^\circ$, $AB < 90^\circ$, $AE > 90^\circ$, and draw the third sides DB, DC, DE. The point C is the pole of the side AD (232.2), and the angle CDA is of 90° , when the opposite side AC is of 90° ; the angle ADB acute when the opposite side $AB < 90^\circ$; and the angle ADE obtuse when the opposite side $AE > 90^\circ$.

245. THEOR. 2. When in a right angled triangle the two oblique angles or their opposite sides are of the same kind, the hypotenuse is less than 90° , and when the two oblique angles or their opposite sides are of a different kind, the hypotenuse is greater than 90° .

DEM. 1°. The hypotenuse is less than 90° when the two oblique sides or angles are less than 90° . For, take the triangle AEO (Fig. 61) with a right angle at E, and the sides AE and EO both less than 90° . Produce AE to B so as to have $AB = 90^\circ$, and EO to C so as to have $EC = 90^\circ$. Then A is the pole of the arc BDC, and $AD > AO = 90^\circ$.

2°. The hypotenuse is also less than 90° when the two oblique sides are greater than 90° . For, take the triangle ABD (Fig. 62) to have a right angle at A contained by two sides AB, AD both greater than 90° : produce these two sides to their second meeting E distant 180° from A, where they form another right angle E (233). The two right angled triangles ABD, EBD

EBD have the same hypotenuse BD, which (by the last) is less than 90° for the triangle EBD, as BE and DE are both less than 90° .

3°. The hypotenuse is greater than 90° when the two oblique sides are one greater, the other less than 90° .—For, take the triangle AFG (Fig. 61.) right angle at F, with the side $AF > 90^\circ$ and the side $FG < 90^\circ$; take on AF the part AB of 90° , and from the pole A draw the arc BDC of 90° : Then $AD = 90^\circ$, and the hypotenuse AG of the given triangle AFG is greater than 90° .

4°. In the triangle ABC (Fig. 61) considered as having a right angle at B contained by two sides BA, BC both of 90° , the hypotenuse $AC = 90^\circ$ of the same kind as the two oblique sides BA, BC; because B is the pole of the arc AC the measure of the right angle B (235).

246. COROL. The reciprocal of the theorem follows from the same demonstrations. 1°. When the hypotenuse is an arc of 90° , the two oblique sides and angles are both of 90° . 2°. When the hypotenuse is less than 90° , the two oblique sides or angles are of the same kind, both less or both greater than 90° . 3°. When the hypotenuse is greater than 90° , the two oblique sides or angles are one greater, the other less than 90° .

247. REMARK. Spherical trigonometry calculates the unknown angles or sides of a triangle by sines, tangents, secants, &c. which are the same for any arc or angle and for its supplement (193). Therefore, if the three given things of a right angled triangle were

were the right angle with either of the two oblique angles and its opposite side, to calculate the hypotenuse and the other oblique angle and side; the sine, tangent or secant of the hypotenuse might be equally applied to an arc less than 90° or to its supplement; and the case should be dubious. So speculative trigonometry leaves dubious cases; but in practical trigonometry, the state of the question generally resolves the doubt. For instance, in Astronomy, if the declination of the sun be given with the obliquity of the Ecliptic, and the sun's longitude or right ascension be required; the hypotenuse, which is the longitude, and the other oblique side, which is the right ascension are known to be less or greater than 90° from the season of the year.

248. THEOREM 3. In the right angled spherical triangle ABC (Fig. 63.) in which B is the right angle; if the three sides be produced to 90° , BA to K and to G, BC to D and to P, AC to E and to F, so that AG, BK, BD, CP, AE, CF are all 90° , and from A and C the two oblique angles as poles be described, the two arcs KP, DG; two new right angled triangles AFK, CED are formed which are the *complementary* triangles of ABC; that is, their angles and sides are the complements of the sides and angles of the triangle ABC.

DEM. By the supposition A is the pole of GD, C of PK, D of GK, K of PD. Therefore the angle D measured by BG is the complement of the oblique side AB; and the angle K measured by BP is the complement of the other oblique side CB. Likewise,
DE

DE is the complement of the angle A, measured by GE; and the side KF is the complement of the angle C measured by PF. Lastly CE or AF is the complement of the hypotenuse AC; AK or BG the complement of the oblique side AB, and CD or BP the complement of the other oblique side CB. Whence the two triangles DEC, AFK right angled at E and F are the complementary triangles of ABC.

The three sides of ABC are supposed to be each less than 90° . If its two oblique sides BA, BC were produced by D and K to their second meeting 180° from B, they should make there another right angle suppose M, and another right angled triangle AMC under the same hypotenuse AC, of which the two oblique sides AM, CM should be each greater than 90° and the supplements of AB and CB, with the same sines, tangents, &c. Of course, the same complementary triangles CED, AFK could be applied to the sides and angles of AMC as well as to ABC. Whence in the usage of sines, tangents and secants, the excess of an angle or arc above 90° may be taken for its complement, as well as what it wants of 90° .

The same demonstration is applicable to the triangle ABC (Fig. 64.) having a right angle at B and the oblique sides, AB less, BC greater than 90° . So every right angled triangle has its complementary triangles whether its oblique sides or angles are of the same or of different kinds.

THEOR.

249. THEOREM 4. In any right angled triangle

Analogy 1. $\left\{ \begin{array}{l} \text{The radius is} \\ \text{to the sine of the hypotenuse} \\ \text{as the sine of either of the oblique angles} \\ \text{is to the sine of the opposite oblique side} \end{array} \right.$

DEM. Suppose C (Fig. 65) to be the centre of the sphere; CB, CA, CD three radii of it; BAD a spherical triangle right angle at A; its side AB in the plane ACB, AD in the plane ACD, BD in the plane BCD: these three arcs are the measures of their respective plane angles ACB, ACD, BCD. On any point E of AC the intersection of the two perpendicular planes BAC, DAC erect a perpendicular FE; and imagine a plane triangle FEP perpendicular to BC the intersection of the other two planes ACB, DCB. Then the line FE in the plane ACD is perpendicular to the line EP in the plane ACB; and the triangle FEP is right angled at E; Whence, the proportion $FE : FP :: \sin. FPE : R$ (214). The right angled triangle FEC gives $FC : FE :: R : \sin. FCE$. Then by multiplication, $FE \times FC : FP \times FE :: R \times \sin. FPE : R \times \sin. FCE$. Whence, $FC : FP :: \sin. FPE : \sin. FCE$. The triangle CPF has also a right angle at P, and $EC : FP :: R : \sin. FCP$. From the two last proportions $R : \sin. FCP :: \sin. FPE : \sin. FCE$, which is the analogy of the theorem. For, $\sin. FCP = \sin. BCD = \sin. BD$ the hypotenuse of ABD. $\sin. FPE = \sin. ABD$ (234) and $\sin. FCE = \sin. ACD = \sin. AD$, and the last proportion becomes as above $R : \sin. BD :: \sin. ABD : \sin. AD$.

The same may be demonstrated of the angle
X ADB

ADB and its opposite side AB by the triangles Epf and fpC by substituting in the above demonstration p instead of P and f instead of F to obtain a similar proportion $R : \sin. BD :: \sin. ADB : \sin AB$.

250. THEOREM 5. In any right angled spherical triangle

Analogy 2. $\left\{ \begin{array}{l} \text{The radius is} \\ \text{to the sine of either of the oblique} \\ \text{sides} \\ \text{as the tangent of its contiguous ob-} \\ \text{lique angle is} \\ \text{to the tangent of the other oblique} \\ \text{side.} \end{array} \right.$

DEM. With the same construction (Fig. 65) as in the last. The right angled plane triangle ECP gives $EP : EC :: \sin. ECP : R$. and the right angled plane triangle FEC gives $EC : EF :: R : \tan. ECF$. Multiplying the two proportions, comes $EP : EF :: \sin. ECP : \tan. ECF$. The plane triangle FEP gives $EP : EF :: R : \tan. EPF$. Whence,

$R : \tan. EPF :: \sin. ECP : \tan. ECF$; or $R : \sin. ECP :: \tan. EPF : \tan. ECF$; which, by substituting as in the last, becomes the proportion of analogy 2. $R : \sin. AB :: \tan. ABD : \tan. AD$.

The same demonstration, by substituting p instead of P , and f instead of F , will give a similar analogy for the other oblique angle and side ADB and AD; viz. $R : \sin. AD :: \tan. ADB : \tan. AB$.

The two above analogies, with the above properties of right angled spherical triangles are sufficient for the solution of all possible cases of right angled spherical triangles, any two things being given besides the right

right angle. All the cases are 30 in number, as there are obviously 10 combinations two and two of the 5 things the two oblique angles and the 3 sides, and three cases to be resolved with each of the 10. However the 30 cases may be reduced to 16 as 14 of them are similar. Of the 30 cases 12 are resolved immediately by the two above analogies; 18 require the intermediate application of the complementary triangles. A *Table* shall be annexed, containing the 16 analogies of the 16 different cases applied to the 30 solutions, with the derivation of the 18 from the complementary triangles, the exceptions and dubious cases alluded to (247).

PROPERTIES

Of Oblique Angled Spherical Triangles.

An *Oblique angled* spherical triangle is that which has no right angle. The peculiar properties and principles for the resolution of oblique angled spherical triangles are the five following analogies.

251. THEOR. 1. In a spherical triangle

Analogy 1. { The sines of the angles are
 { as the sines of the opposite sides.

DEM. Let ABD (Fig. 66, 67) be two oblique angled spherical triangles. From any angle, A for instance, draw an arc AP perpendicular to the opposite BD produced, if necessary, as in (Fig. 67), to divide the oblique angled triangle ABD into two Right

X 2

angled

angled triangles APB, APD, the right angles at P. Then by the analogy (249)

$$\left. \begin{array}{l} R : \sin. AB :: \sin. B : \sin. AP \\ R : \sin. AD :: \sin. D : \sin. AP \end{array} \right\} \text{Whence}$$

$$\sin. B : \sin. D :: \sin. AD : \sin. AB.$$

The perpendicular arc AP being drawn from any other angle, D for instance, on the opposite side AB, the demonstration should give $\sin. A : \sin. B :: \sin. BD : \sin. AD$. Therefore $\sin. A : \sin. BD :: \sin. B : \sin. AD :: \sin. D : \sin. AB$.

252. COROL. Hence again as (240) an Equilateral triangle is also equiangular, and reciprocally. Because, equal sines belong to equal angles and to equal arcs, and reciprocally.

253. THEOREM 2. A perpendicular arc being drawn from any angle to the opposite side,

Analogy 2. $\left\{ \begin{array}{l} \text{The cosines of the two segments are} \\ \text{as the cosines of their contiguous sides} \end{array} \right.$

DEM. The complementary triangle CED (Fig. 63) Right angled at E, gives the analogy $R : \sin. CD :: \sin. D : \sin. CE$ (249.) which transferred to the triangle ABC becomes $R : \cos. BC :: \cos. AB : \cos. AC$ (248); and this last applied to the two right angled triangles APB, APD, (Fig. 66, 67) gives on APB, $R : \cos. BP : \cos. AP : \cos. AB$, and on APD, $R : \cos. DP :: \cos. AP : \cos. AD$. From these two proportions, $R \times \cos. AB : R \times \cos. AD :: \cos. AP \times \cos. BP : \cos. AP \times \cos. DP$; Whence, $\cos. BP : \cos. DP :: \cos. AB : \cos. AD$.

254. COROL. The cosines of the two angles BAP, DAP made by the perpendicular arc AP with the

the two sides AB, AD are directly as the cotangents of the two respective sides.

DEM. In the complementary triangle AFK (Fig. 63) right angle at F, $R : \sin. KF : \text{tang. } K : \text{tang. } AF$ (250), which on ABC becomes $R : \cos. BCA :: \cot. BC : \cot. AC$; and this applied to (Fig. 66, 67) gives, on BAP, $R : \cos. BAP :: \cot. AP : \cot. AB$; and on DAP, $R : \cos. DAP :: \cot. AP : \cot. AD$. Whence, $\cos. BAP : \cos. DAP :: \cot. AB : \cot. AD$.

255. THEOREM 3. With the same perpendicular arc AP.

Analogy 3. $\left\{ \begin{array}{l} \text{The sines of the two segments, are} \\ \left\{ \begin{array}{l} \text{directly as the cotan-} \\ \text{gents inversely as the} \\ \text{tangents} \end{array} \right\} \text{ of the two} \\ \text{contiguous} \\ \text{angles B \& D} \end{array} \right.$

DEM. By the analogy (250) we have

$\left. \begin{array}{l} R : \sin. DP :: \text{tang. } D : \text{tang. } AP \\ R : \sin. BP :: \text{tang. } B : \text{tang. } AP \end{array} \right\} \begin{array}{l} \text{whence } \sin. BP : \sin. DP \\ :: \text{tang. } D : \text{tang. } B \end{array}$
and as the tangents of two arcs are reciprocally as their cotangents (196.) $\sin. BP : \sin. DP :: \cot. B : \cot. D$.

256. COROL. The tangents of the two angles BAP, DAP are directly as the tangents of their opposite segments.

DEM. By the Analogy (250)

$\left. \begin{array}{l} R : \sin. AP :: \text{tang. } BAP : \text{tang. } BP \\ R : \sin. AP :: \text{tang. } DAP : \text{tang. } DP \end{array} \right\} \begin{array}{l} \text{tang. } BAP : \text{tang. } DAP \\ :: \text{tang. } BP : \text{tang. } DP. \end{array}$

257. THEOREM 4. With the same perpendicular arc AP.

Analogy 4. $\left\{ \begin{array}{l} \text{tang. } \frac{BD}{2} : \text{tang. } \frac{AB + AD}{2} :: \\ \text{tang. } \frac{AB - AD}{2} : \text{tang. } \left\{ \begin{array}{l} \frac{BP - DP}{2} \text{ (Fig. 66)} \\ \frac{BP + DP}{2} \text{ (Fig. 67)} \end{array} \right. \end{array} \right.$

DEM. Plane Trigonometry demonstrates (199) that the sum of the sines of two arcs is to their difference, as the tangent of half the sum of the same two arcs is to the tangent of half their difference; and also that the sum of the cosines of two arcs is to their difference, as the cotangent of half the sum of the same two arcs is to the tangent of half their difference. Then in the present case

$$\left\{ \begin{array}{l} \cos. AD + \cos. AB : \cos. AD - \cos. AB :: \\ \cot. \frac{AB + AD}{2} : \text{tang. } \frac{AB - AD}{2} \\ \cos. DP + \cos. BP : \cos. DP - \cos. BP :: \\ \cot. \frac{BP + DP}{2} : \text{tang. } \frac{BP - DP}{2} \end{array} \right\} \text{But}$$

$\cos. AD : \cos. AB :: \cos. DP : \cos. BP$ (253), which addendo and subtrahendo gives, $\cos. AD + \cos. AB : \cos. AD - \cos. AB :: \cos. DP + \cos. BP : \cos. DP - \cos. BP$. Therefore the two first ratios of the two

first proportions are equal, and $\cot. \frac{AB + AD}{2} : \text{tang. } \frac{AB - AD}{2} :: \cot. \frac{BP + DP}{2} : \text{tang. } \frac{BP - DP}{2}$ and chang-

ing the middle terms, the proportion becomes

$$\cot. \frac{AB + AD}{2} : \cot. \frac{BP + DP}{2} :: \text{tang. } \frac{AB - AD}{2} :$$

tang.

$\text{tang.} \frac{BP - DP}{2}$. But the tangents are reciprocally proportional to the cotangents (196). Therefore substituting the tangents, $\text{tang.} \frac{BP + DP}{2} ; \text{tang.} \frac{AB + AD}{2} :: \text{tang.} \frac{AB - AD}{2} ; \text{tang.} \frac{BP - DP}{2}$; and reading this proportion backwards and changing the middle terms $\text{tang.} \frac{BP - DP}{2} ; \text{tang.} \frac{AB + AD}{2} :: \text{tang.} \frac{AB - AD}{2} ; \text{tang.} \frac{BP + DP}{2}$; then uniting the two proportions into one by a double sign to the extremes,

$$\text{tang.} \frac{BP \pm DP}{2} = \text{tang.} \frac{BD}{2} : \text{tang.} \frac{AB + AD}{2} :: \text{tang.} \frac{AB - AD}{2} : \text{tang.} \begin{cases} \frac{BP - DP}{2} \text{ (Fig. 66)} \\ \frac{BP + DP}{2} \text{ (Fig. 67)} \end{cases}$$

258. THEOREM 5. In any spherical triangle CAD (Fig. 68) we have the following

$$\text{ANALOGY 5. Sin. AC} \times \text{sin. AD} : \text{sin.} \frac{AC + CD + AD}{2} - AC \times \text{sin.} \frac{AC + CD + AD}{2} - AD :: R^2 : \text{sin.} \frac{1}{2} CAD^2$$

That is, the product of the sines of the two sides containing the angle CAD : is to the product of the sines of the differences of each of the same two sides with half the sum of the three sides, as the square of the Radius is to the square of the sine of half the angle CAD.

DEM.

DEM. Suppose HAK to be a semi-great circle of a Sphere, the centre of which is O, and HLK another semi-great circle, the plane of which is perpendicular to the plane of the first, (HAK). Draw the radius of the sphere AO perpendicular to HK the intersection of the two planes. From the angle A of the given spherical triangle CAD describe a small circle of the sphere BDE by the vertex of the angle D, which has its plane perpendicular to the plane of the great circle HAK and parallel to the plane of HLK, the chord BE being its Diameter. From the angle C describe another small circle of the sphere FDG, having also its plane perpendicular to the plane of HAK, and the chord FG its diameter.

From this construction; A being the centre of the small circle BDE, $AD = AE = AB$; and C being the centre of FDG, $CD = CF = CG$; Whence the arc $FE = FC + CA + AE = CD + CA + AD$ the sum of the three sides of the given spherical triangle ACD; and $CB = AB - AC = AD - AC$ the difference of the two sides containing the angle A.

Likewise, $BG = CG + CB = CD + CB$; and as $CB = AD - AC + AC - AC = AD + AC - 2AC$, $BG = CD + CB = CD + AD + AC - 2AC$; Whence $\frac{1}{2} BG = \frac{CD + AD + AC}{2} - AC$.

And $BF = CF - CB = CD - CB$; and as $CB = AD - AC - AD + AD = 2AD - AC - AD$, $BF = CD - CB = CD + AD + AC - 2AD$; Whence $\frac{1}{2} BF = \frac{CD + AD + AC}{2} - AD$

Produce

Produce now to L the side AD of the triangle ACD ; and the arc HL of which A is the pole (232. 2) will be the measure of the angle CAD (235) and HQ the half of the chord HL is the sine of half the same angle CAD (194) and so is one half of the chord BF the sine of half the angle BIF . Draw the radius of the sphere OQM , which, as it bisects the chord HL and the arc sustained by it is perpendicular on HL . Draw LP and QN both perpendicular to the diameter HK , then LP is the sine of the arc LK , the sine of its supplement LH , the sine of the angle CAD . Draw OR parallel to FG , the angles $BIF = ROK$; and as $CG = CF$, the radius CO perpendicular on FG is perpendicular on OR , the arc $CR = 90^\circ = AK$, and $RK = AC$. Draw DI the intersection of the planes of the two circles BDE , FDG ; it is perpendicular to BE , because the two planes are perpendicular to the plane of the great circle HAK (152. 2). The two arcs DE , LK are of the same number of degrees (232. 6); Whence LP and DI are homologous ordinates of the two circles, and divide their diameters into homologous proportional segments. Lastly, BS is the sine of the arc AB , the sine of the side AD .

Now, in the triangle BIF the sines of the angles are as their opposite sides (197); $\sin. BIF = \sin. ROK = \sin. RK = \sin. AC : \sin. BFI = \sin. \frac{1}{2} BG :: BF : BI :: \frac{1}{2} BF : \frac{1}{2} BI$; or (a) $\sin. AC : \sin. \frac{1}{2} BG :: \frac{1}{2} BF : \frac{1}{2} BI$.

The diameters BE , HK are proportional to their homologous segments; $BI : HP :: BE : HK$; and

Y

$\frac{1}{2} BI$

$$\frac{1}{2} BI : \frac{1}{2} HP = HN :: \frac{1}{2} BE = BS : \frac{1}{2} HK = HO ; \text{ or } (b) \\ BS : HO :: \frac{1}{2} BI : HN$$

The two triangles HQO, HNQ are similar, as they have a common angle H and a right angle at Q and N; Whence, $\div$ HO : HQ : HN; and (d) HO : HN :: HO² : HQ²

Multiply the three above proportions (a, b, d) term by term, and

$$\text{divide in the product one ratio by HO} \left\{ \begin{array}{l} (a) \sin. AC : \sin. \frac{1}{2} BG :: \frac{1}{2} BF : \frac{1}{2} BI \\ (b) \quad BS : \quad HO :: \frac{1}{2} BI : HN \\ (d) \quad HO : \quad HN :: HO^2 : HQ^2 \end{array} \right\} \text{ Comes}$$

Sin. AC $\times$ BS : sin $\frac{1}{2}$ BG $\times$ HN :: $\frac{1}{2}$ BF $\times$ HO² ; HN $\times$ HQ². An equation being made of the products of

the extremes and means and the common factor HN exterminated, a new proportion arises sin. AC $\times$ BS

: sin. $\frac{1}{2}$ BG $\times$ $\frac{1}{2}$ BF :: HO² : HQ². Substitute in this

$$\text{the values found above; viz. } BS = \sin. AD ; \frac{1}{2} BG = \frac{CD + AD + AC}{2} - AC ; \frac{1}{2} BF = \frac{CD + AD + AC}{2} - AD ;$$

HO = R ; HQ = sin. $\frac{1}{2}$ CAD, and the last proportion becomes the above stated Analogy 5.

$$\sin. AC \times \sin. AD : \left(\sin. \frac{CD + AD + AC}{2} - AC \right) \times \left(\sin. \frac{CD + AD + AC}{2} - AD \right) :: R^2 : \sin. \frac{1}{2} CAD^2$$

The above analogies and properties of the oblique angled spherical triangles are sufficient for the resolution of all the cases of such triangles. All the cases are 12 in number; 8 of them require the perpendicular arc from one of the angles to the opposite side, in

in order to divide the triangle into two right angled triangles; that perpendicular arc falls within the triangle when the two other angles are of the same kind; it falls outside when they are of a different kind. Sometimes the help of the supplementary triangles is also necessary (238). The 12 cases may, however, be reduced to 6, as the solutions are similar two and two, as will be seen hereafter.

Remains now to subjoin to the above theory a few solutions of problems on right angled and oblique-angled spherical triangles.

RESOLUTION

Of Right Angled Spherical Triangles.

There are, as was said above, 30 cases of right angled triangles, which are all resolved by the two analogies (249. 250); But the immediate application of the analogies resolves only some; for the rest, the analogies are applied to the complementary triangles and transferred. A table is subjoined, containing the 30 analogies for the 30 cases. It consists of 5 columns; the first is the three data for the problem; the 2d. the three parts of the triangle to be calculated; the 3d. the analogy for every corresponding case on the triangles ABC (Fig. 63) right angle at B; the 4th the analogy taken on the complementary triangles CED, AFK, and transferred to the third column on the triangle ABC; the 5th. the exceptions and dubious cases, for which it is to be remarked that the

triangle ABC is supposed to have its oblique sides and angles less than 90° .

259. PROBLEM 1. Suppose the triangle ABC (Fig. 63) to have the oblique angle $C = 23^\circ 28' 50''$ and the opposite side $AB = 11^\circ 44''$; to find the hypotenuse AC?

SOL. This is the 11th case in the table solved by the immediate application of the analogy (249).

Sin. C : sin. AB :: R : sin. AC. By Logarithms

$$\left\{ \begin{array}{l} 9, 30826 \quad \text{Log. sin. } 11^\circ 44' = AB \\ - 9, 60012 \quad \text{Log. sin. } 23^\circ 28' = C \end{array} \right\} =$$

$$9, 70814 \quad \text{Log. sin. } AC = 30^\circ 42' 30''.$$

This solution, in Astronomy, gives the Longitude of the Sun AC, its declination AB being supposed $11^\circ 44'$ the half of the greatest declination, and the angle C, the inclination of the Ecliptic to the Equator $23^\circ 28'$ nearly.

260. PROBLEM 2. Suppose the triangle ABC (Fig. 63) to have the oblique angle $C = 23^\circ 28'$ as above, and the oblique side $BC = 45^\circ$; to find the other oblique angle A?

SOL. This is the 19th case in the table solved by the analogy (249) applied to the complementary triangle CED, which transferred on ABC becomes $R : \cos. BC :: \sin. C : \cos. A = \sin. C \times \cos. BC$. By Logarithms

$$\left\{ \begin{array}{l} 9,60012. \quad \text{Log. sin. } 23^\circ 28' = C \\ + 9,84949. \quad \text{Log. cos. } 45^\circ = BC \end{array} \right\} =$$

$$9,44961. \quad \text{Log. cos. } 73^\circ 21' 40'' = A$$

This solution may be used in Astronomy to calculate the relative obliquity of the orbits of the planets or satellites

satellites to the equator, or to the orbit of the principal planet.

261. PROBLEM 3. Suppose the same ABC to have the angle $C = 36^\circ 42'$, and the side $AB = 23^\circ 28'$ to find AC?

SOL. This is also the 11th case of the table : $\sin. C : \sin. AB :: R : \sin. AC$. By Logarithms,

$$\left\{ \begin{array}{l} 9,60012. \quad \text{Log. sin. } 23^\circ 28' = AB \\ -9,77643. \quad \text{Log. sin. } 36^\circ 42' = C \end{array} \right\} =$$

$$9,82369 \quad \text{Log. sin. } 41^\circ 47' = AC$$

This solution calculates the amplitude of the Sun rising and setting on the solstitial days at the Latitude $53^\circ 18'$.

RESOLUTION

Of Oblique Angled Spherical Triangles.

The analogies from (251) to (258) resolve the 12 possible cases of oblique angled spherical triangles. The cases will be supposed on the (Fig 66, 67) and on the (Fig. 57) when the supplementary triangles are made use of.

262. CASE 1. (Fig. 66.) Being given AB and AD with the angle B opposite to one of the two given sides, to find the angle D opposite to the other given side?

SOL. The analogy 1 (251) gives $\sin. AD : \sin. AB :: \sin. B : \sin. D$.

263. CASE 2. Being given, the two angles B and D with the side AD opposite to either of the two angles,

angles, to find the side AB opposite to the other given angle?

SOL. The same analogy gives $\sin. B : \sin. D :: \sin. AD : \sin. AB$.

264. CASE 3. Being given, two sides AB and AD with the angle D opposite to either of the two given sides, to find the third side BD (Fig. 66, 67)?

SOL. Supposing the arc AP perpendicular, we know in the right angled triangle APD the oblique angle D and the hypotenuse AD; the other oblique side the segment DP is found by the table of right angled triangles (Case 27) the analogy of which applied to APD becomes $\cos. D : R :: \cot. AD : \cot. DP$, or taking the tangents reciprocal to the cotangents (196) $R : \cos. D :: \text{tang. AD} : \text{tang. DP}$. Which gives the segment DP.

The other segment BP is given by the analogy (253) $\cos. AD : \cos. AB :: \cos. DP : \cos. BP$. Whence the third side

$$BD = \begin{cases} BP + DP & \text{when the two given sides are of the} \\ & \text{same kind.} \\ BP - DP & \text{when the two given sides are of a} \\ & \text{different kind.} \end{cases}$$

265. CASE 4 (Fig 57) Being given two angles A and B with the side BD opposite to A, to find the third angle D?

SOL. Take in the supplementary triangle FEG the supplements of the three given things EG, FG, F (238) and find the third side EF as (264.) that side EF is the supplement of the angle D.

266. CASE 5. (Fig. 66. 67.) Being given two angles B and D with one of their opposite sides AB, to find the side BD between the two given angles?

SOL. Supposing the arc AP perpendicular on the side BD, the two segments of BD are found by these proportions,

$$\left. \begin{array}{l} R : \cos. D :: \text{tang. AD} : \text{tang. DP (the same as 264)} \\ \cot. D : \cot. B :: \sin. DP : \sin. BP \text{ (255)} \end{array} \right\}$$

$$BP \pm DP = BD$$

267. Case 6. (Fig. 57.) Being given, two sides AB and AD with the angle B opposite to either of the two given sides, to find the angle A contained by the two given sides?

SOL. Take in the supplementary triangle FEG the supplements of the three given things G, E, FG (238). Then knowing two angles and one of their opposite sides in the triangle FEG, the side EG between the two angles G, E, is found as above (266) and that side EG is the supplement of the angle A.

268. CASE 7. (Fig. 66, 67.) Being given two sides AB and BD with the angle B contained by the two given sides, to find the third side AD?

SOL. Draw the perpendicular arc AP, and find the segment BP as (264) $R : \cos. B :: \text{tang. AB} : \text{tang. BP}$: take the difference between the given side BD and that segment BP, to find the other segment DP. Then to find AD make the proportion $\cos. BP : \cos. DP :: \cos. AB : \cos. AD$ (253).

269. CASE 8. (Fig. 57) Being given two angles A and D with the side AD between the two given angles, to find the third angle B?

SOL.

SOL. In the supplementary triangle EFG take the supplements of the three given things EG, FE, E (238). Then knowing the two sides EG, FE of FEG with the angle E contained by them, find the third side FG as (268); it is the supplement of the angle B.

270. CASE 9. (Fig. 66, 67.) Being given two sides AB and BD with the angle B contained by the two given sides, to find the angle D?

SOL. With the perpendicular arc AP find the segments BP, DP as (268); then find the angle D by the following proportion $\sin. BP : \sin. DP :: \cot. B : \cot. D$ (255).

271. CASE 10. (Fig. 57) being given two angles A and D with the side AD contained by the two given angles, to find the side AB opposite to the given angle D?

SOL. Take in the supplementary triangle FEG the supplements of the three given things EG, FE, E; which being known, calculate the angle G opposite to FE as (270), it is the supplement of the side AB.

272. CASE 11. (Fig. 66, 67) Being given the three sides AB, BD, AD, to find one of the three angles, B?

SOL. 1. With the perpendicular arc AP from the angle A to the side BD contiguous to the angle B, knowing the sum or the difference of the two segments equal to the side BD, and of course, half their sum or half their difference, find half their difference or half their
sum

sum by the analogy (257) $\text{tang. } \frac{BD}{2} : \text{tang. } \frac{AB + AD}{2}$

$$:: \text{tang. } \frac{AB - AD}{2} : \text{tang. } \begin{cases} \frac{BP - DP}{2} \text{ (Fig. 66)} \\ \frac{BP + DP}{2} \text{ (Fig. 67)} \end{cases} \quad \text{that half}$$

difference or half sum added to half the sum or difference $\frac{BD}{2}$ gives the greater segment BP; and you know in the right angled triangle APB the oblique side BP and the hypotenuse AB. With such data, the angle B is found by this analogy in the table (case 14).

$$\begin{aligned} \cot. BP : \cot. AB :: R : \cos. B \text{ or (196) } \text{tang. } AB : \\ \text{tang. } BP :: R : \cos. B. \end{aligned}$$

SOL. 2. Another solution of the same case is given by the analogy (258) which for the angle B becomes

$$\begin{aligned} \sin. AB \times \sin. BD : \sin. \frac{AB + BD + AD}{2} - AB \times \\ \sin. \frac{AB + BD + AD}{2} - BD :: R^2 : \sin. \frac{1}{2} ABD^2 \end{aligned}$$

This second solution is more convenient and more expeditious in practical trigonometry.

273. CASE 12. (Fig. 57). Being given the three angles A, B, D, to find any of the sides, AD?

SOL. Take the supplements of the three given angles: they are the three sides of the supplementary triangle FEG; which three sides being known, the case consists in calculating one of the angles E as

Z

in

in (Case 11). This last case is of very little use in practical trigonometry.

The following table of the 12 cases of oblique angled spherical triangles is only a reference to the analogy which resolves every particular case.

Table for the Resolution of Oblique angled Spherical Triangles.

Case	Given	To find	Analogy found.	Case	Given	To find	Analogy found.
1	AB, AD, B Fig. 66. 67.	D	No. 262	7	AB, B, B Fig. 66. 67	AD	No. 268
2	B, D, AD Fig. 66. 67	AB	263	8	A, D, AD Fig. 57	B	269
3	AB, AD, D Fig. 66. 67	BD	264	9	AB, BD, B Fig. 66. 67	D	270
4	A, B, D Fig. 57	D	265	10	A, D, AD Fig. 57	AB	271
5	B, D, AB Fig. 66. 67	BD	266	11	AB, BD, AD Fig. 66. 67	B	272
6	AB, AD, B Fig. 57	A	267	12	A, B, D, Fig. 57	AD	273

274. PROBLEM. Being given the three sides (Fig. 66. 67.)

Case 11. Analogy (272) to find the angle ABD

$$AD = 62^{\circ} 10'$$

$$AB = 59^{\circ} 30' \quad \text{Log. sin. } 9,93532$$

$$BD = 74^{\circ} 42' \quad \text{Log. sin. } 9,98433 \quad \left. \vphantom{\begin{matrix} AD \\ AB \\ BD \end{matrix}} \right\} = \dots - 19,91965$$

$$\text{Sum } 196^{\circ} 22'$$

$$\frac{1}{2} \text{ Sum } 98^{\circ} 11'$$

$$\frac{1}{2} \text{ Sum } - 59^{\circ} 30' = 38^{\circ} 41' \quad \text{Log. sin. } 9,79589$$

$$\frac{1}{2} \text{ Sum } - 74^{\circ} 42' = 23^{\circ} 29' \quad \text{Log. sin. } 9,60041 \quad \left. \vphantom{\begin{matrix} \frac{1}{2} \text{ Sum} \\ \frac{1}{2} \text{ Sum} \end{matrix}} \right\} = + 19,39630$$

$$+ 20 = 2 \text{ Log. R}$$

$$Df = 19,47665$$

$$\frac{1}{2} \text{ Dif. } = 9,73833 \text{ L. sin } \frac{1}{2} B$$

The last Logarithm 9,73833 is the Log. of $33^{\circ} 11' 30''$ the double of which is the angle required $ABD = 66^{\circ} 23'$

Such a problem is very frequent in Astronomy, particularly for Navigation; When the hour of the day is to

to be found from the vertical elevation of the sun, or any other celestial body. In the present case B is supposed to be the pole of the celestial sphere, A the zenith of the place, AB the complement of the latitude, AD the complement of the vertical altitude of the sun, BD the complement of the sun's declination.

The horary angle $66^{\circ} 23'$ reduced into time gives 4 hours 25 minutes and 31 seconds, which subtracted from 12 hours leaves 7h. 34m. 29s. the hour of the day.

270	D	AD, BD, A	AB, AD, B	BD, AD, B	AB, BD, A
271	AB	A, D, AD	A, B, AB	B, D, BD	B, A, AB
272	B	A, B, AB	A, D, AD	B, D, BD	A, B, AB
273	AD	A, D, AD	A, B, AB	B, D, BD	A, B, AB

FINIS.

Case II. Analogy (272) to find the angle ABD. (Fig. 66. 67.)

$$\begin{aligned}
 & \text{AD} = 62^{\circ} 10' \\
 & \text{AB} = 50^{\circ} 30' \quad \text{Log. sin. } 9.93232 \\
 & \text{BD} = 74^{\circ} 42' \quad \text{Log. sin. } 9.98433 \\
 & \text{Sum } 186^{\circ} 22' \\
 & \frac{1}{2} \text{ Sum } 93^{\circ} 11' \\
 & \frac{1}{2} \text{ Sum } - 29^{\circ} 30' = 38^{\circ} 41' \quad \text{Log. sin. } 9.79589 \\
 & \frac{1}{2} \text{ Sum } - 74^{\circ} 42' = 23^{\circ} 29' \quad \text{Log. sin. } 9.60041 \\
 & \text{+ } 20 = 2 \text{ Log. R} \\
 & \text{+ } 10.36630 \\
 & \text{D.C.} = 10.47662 \\
 & \frac{1}{2} \text{ Dif.} = 9.73833
 \end{aligned}$$

The last Logarithm 9.73833 is the Log. of $33^{\circ} 11' 30''$ the double of which is the angle required ABD = $66^{\circ} 23'$

Such a problem is very frequent in Astronomy, particularly for Navigation; When the hour of the day is

ERRATA.

Page.	Line.	Read.
98	last	— as $1\frac{1}{8}$ to 4 instead of <i>to A.</i>
112	5	— sum. instead of <i>sine</i>
—	19	— $\frac{1}{2}$ AD
—	26	— tang $\frac{a+b}{2}$ instead of $\frac{—}{2}$
116	20--21	— of a instead of <i>of A</i>
118	2	— Cos. $b \pm$ sin $b \times$ Cos. a
—	3	— Cos. $b \mp$ sin. a $\times$ sin b.
124	6	— AC—AP. instead of <i>A—CAP</i>
132	8	— $36^{\circ} 30'$ instead of $36^{\circ} 33'$
138	14	<i>erase</i> the two words <i>not wanting</i>
147	10	<i>read</i> BL and DG, instead of <i>BD and DG</i>
162	21	— AC instead of AC^2
164	4	— $23^{\circ} 28'$ instead of $23^{\circ} 28' 50''$
—	5	— 44' instead 44"

fig. 58 in Plate, change B instead of D and D instead of B.

Cases.	Given with B	to find	Analogies on the triangle ABC.	Analogies on the Compt. triang. CED, AFK transferred on ABC.	EXCEPTIONS.
1		C	$\sin. AC : R :: \sin. AB : \sin. C$		
2	AB·AC	A	$\cot. AB : \cot. AC :: R : \cos. A$	$\tan. D : \tan. CE :: R : \sin. DE$	if $\begin{cases} AB < 90^\circ \\ AB, AC \text{ of the same kind} \\ AB, AC \text{ of the same kind} \end{cases}$
3		BC	$\cos. AB : \cos. AC :: R : \cos. BC$	$\sin. D : \sin. CE :: R : \sin. DC$	
4		A	$\sin. AB : R :: \tan. BC : \tan. A$		
5	AB·BC	C	$\sin. BC : R :: \tan. AB : \tan. C$		if $\begin{cases} BC < 90^\circ \\ AB < 90^\circ \\ AB, AC \text{ of the same kind} \end{cases}$
6		AC	$R : \cos. BC :: \cos. AB : \cos. C$	$\begin{cases} R : \sin. DC :: \sin. D : \sin. CE \\ R : \sin. AK :: \sin. K : \sin. AF \end{cases}$ both to find AC	
7		C	$R : \cos. AB :: \sin. A : \cos. C$	$R : \sin. AK :: \sin. A : \sin. KF$	
8	AB·A	AC	$R : \cos. A :: \cot. AB : \cot. AC$	$R : \sin. DE :: \tan. D : \tan. CE$	if $\begin{cases} AB < 90^\circ \\ AB, A \text{ of the same kind} \\ A < 90^\circ \end{cases}$
9		BC	$R : \sin. AB :: \tan. A : \tan. BC$		
10		A	$\cos. AB : R :: \cos. C : \sin. A$	$\sin. AK : R :: \sin. KF : \sin. A$	
11	AB·C	AC	$\sin. C : \sin. AB :: R : \sin. AC$		Dubious
12		BC	$\tan. C : \tan. AB :: R : \sin. BC$		Dubious
13		A	$\sin. AC : R :: \sin. BC : \sin. A$		Dubious
14	BC·AC	C	$\cot. BC : \cot. AC :: R : \cos. C$	$\tan. K : \tan. AF :: R : \sin. KF$	if $\begin{cases} BC < 90^\circ \\ AC, BC \text{ of the same kind} \\ AC, BC \text{ of the same kind} \end{cases}$
15		AB	$\cos. BC : \cos. AC :: R : \cos. AB$	$\sin. K : \sin. AF :: R : \sin. AK$	
16		C	$\cos. BC : R :: \cos. A : \sin. C$	$\sin. DC : R :: \sin. DE : \sin. C$	
17	BC·A	AC	$\sin. A : \sin. BC :: R : \sin. AC$		Dubious
18		AB	$\tan. A : \tan. BC :: R : \sin. AB$		Dubious
19		A	$R : \cos. BC :: \sin. C : \cos. A$	$R : \sin. DC :: \sin. C : \sin. DE$	if $\begin{cases} BC < 90^\circ \\ BC, C \text{ of the same kind} \\ C < 90^\circ \end{cases}$
20	BC·C	AC	$R : \cos. C :: \cot. BC : \cot. AC$	$R : \sin. KF :: \tan. K : \tan. AF$	
21		AB	$R : \sin. BC :: \tan. C : \tan. AB$		
22		C	$\cos. AC : R :: \cot. A : \tan. C$	$\sin. CE : R :: \tan. DE : \tan. C$	if $\begin{cases} AC, A \text{ of the same kind} \\ AC, A \text{ of the same kind} \end{cases}$
23	AC·A	AB	$\cos. A : R :: \cot. AC : \cot. AB$	$\sin. DE : R :: \tan. CE : \tan. D$	
24		BC	$R : \sin. AC :: \sin. A : \sin. BC$		
25		A	$R : \cos. AC :: \tan. C : \cot. A$	$R : \sin. CE :: \tan. C : \tan. DE$	if $\begin{cases} AC, C \text{ of the same kind} \\ C < 90^\circ \\ AC, C \text{ of the same kind} \end{cases}$
26	AC·C	AB	$R : \sin. AC :: \sin. C : \sin. AB$		
27		BC	$\cos. C : R :: \cot. AC : \cot. BC$	$\sin. KF : R :: \tan. AF : \tan. K$	
28		AC	$\tan. C : \cot. A :: R : \cos. AC$	$\tan. C : \tan. DE :: R : \sin. CE$	if $\begin{cases} A, C \text{ of the same kind} \\ C < 90^\circ \\ A < 90^\circ \end{cases}$
29	A·C	AB	$\sin. A : \cos. C :: R : \cos. AB$	$\sin. A : \sin. KF :: R : \sin. AK$	
30		BC	$\sin. C : \cos. A :: R : \cos. BC$	$\sin. C : \sin. DE :: R : \sin. DC$	

TABLE I.

OF THE 30 ANALOGIES

For the Resolution of the 30 Cases of Right-angled Spher-triangles.

B is the Right-angle of the triangle ABC

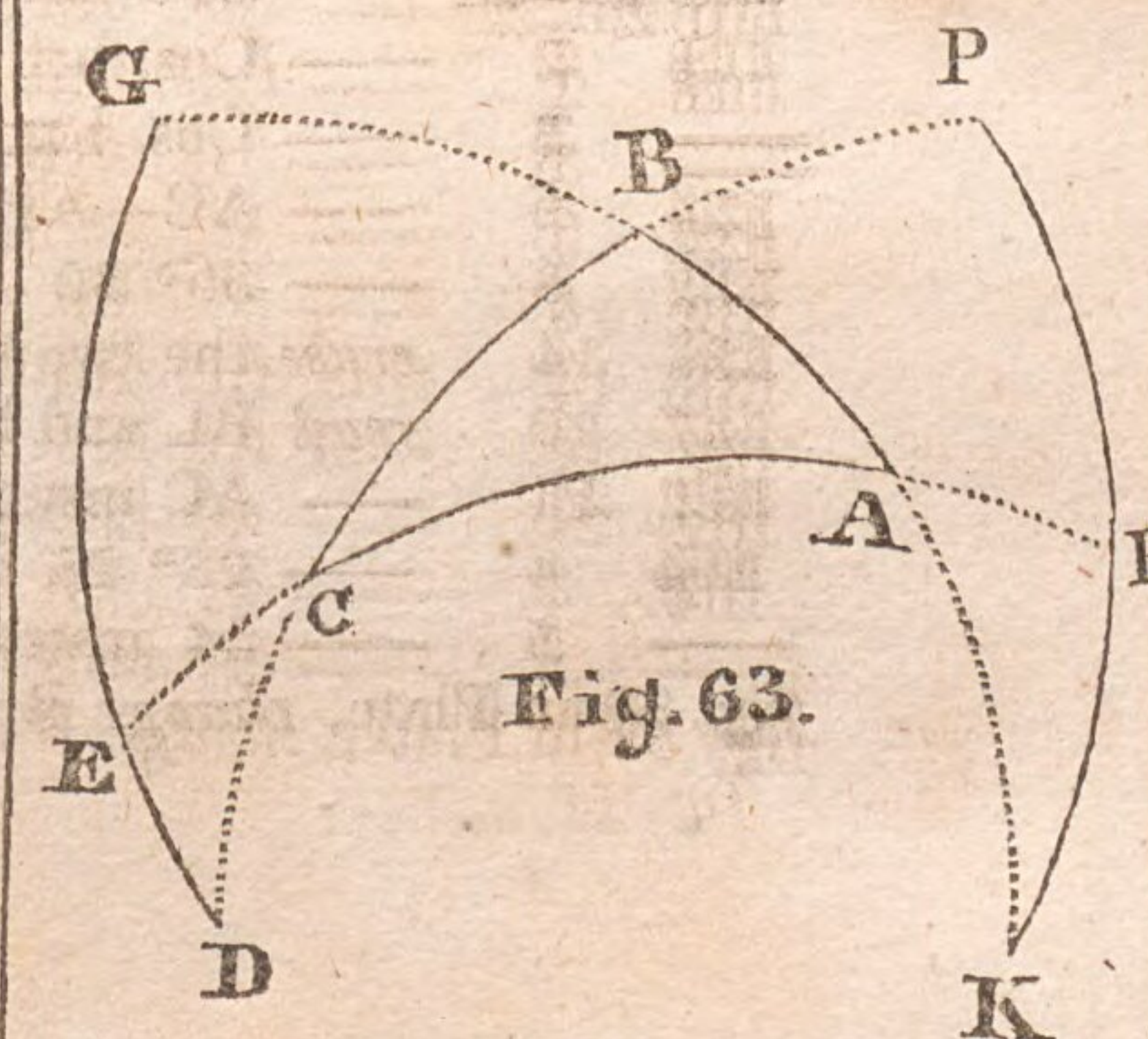
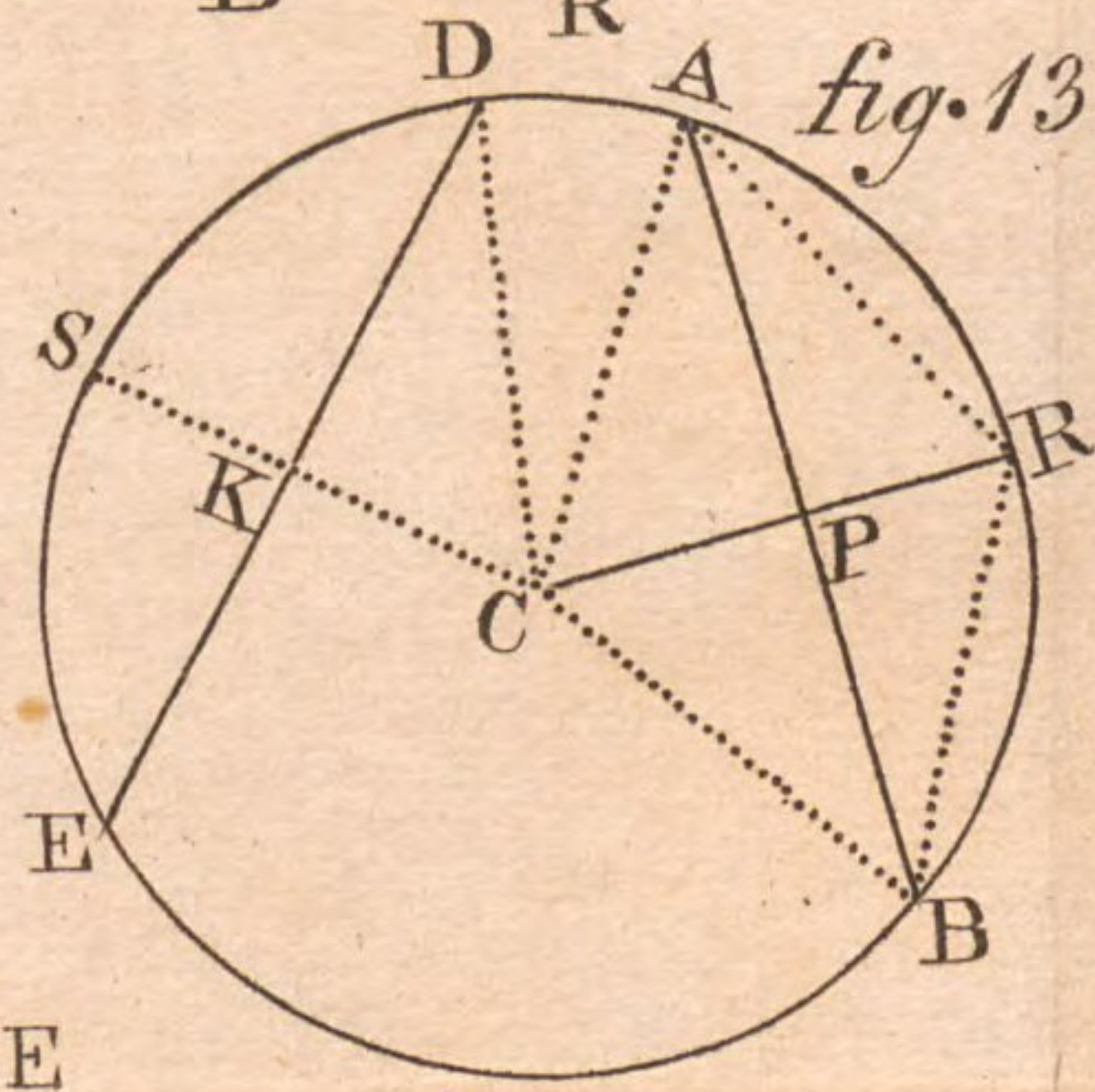
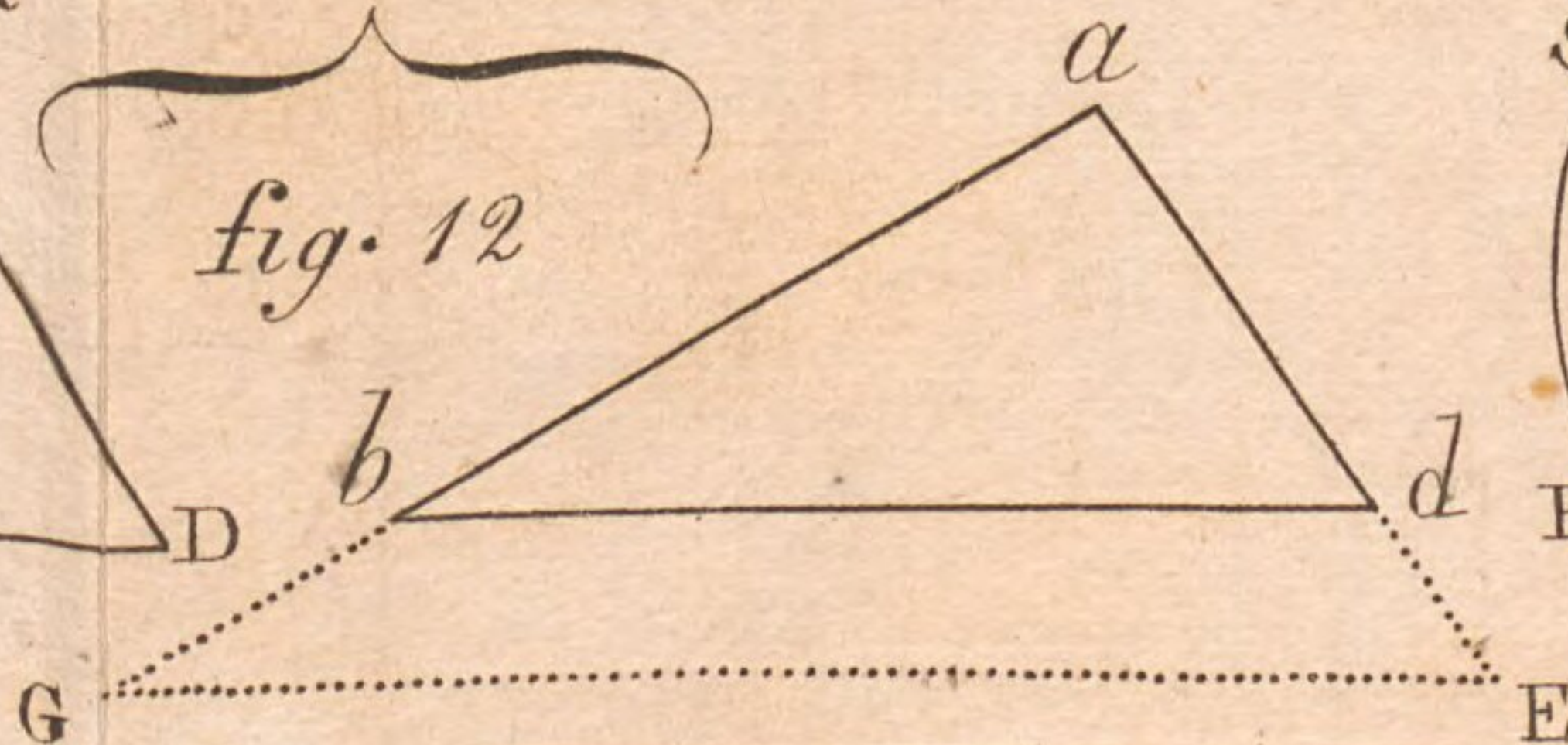
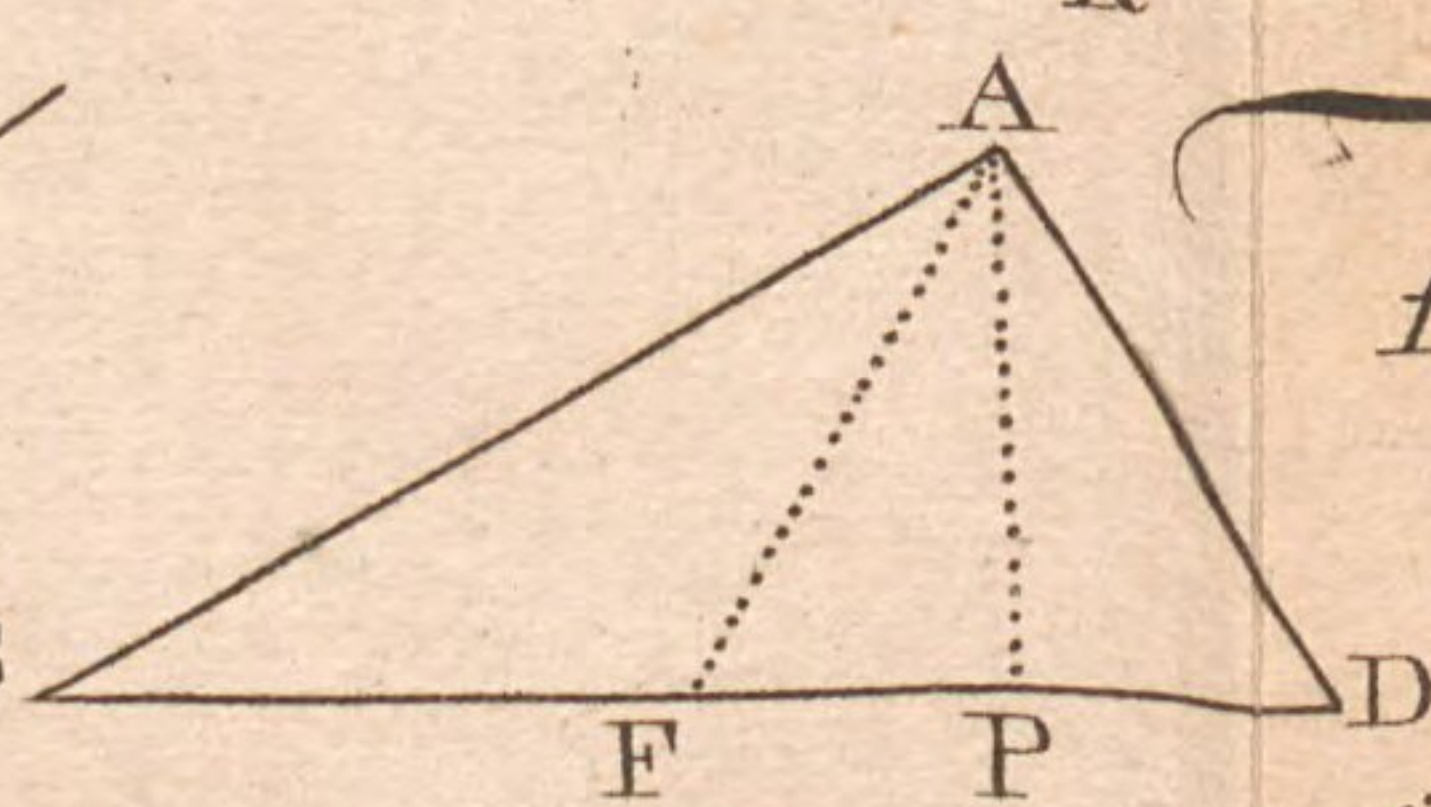
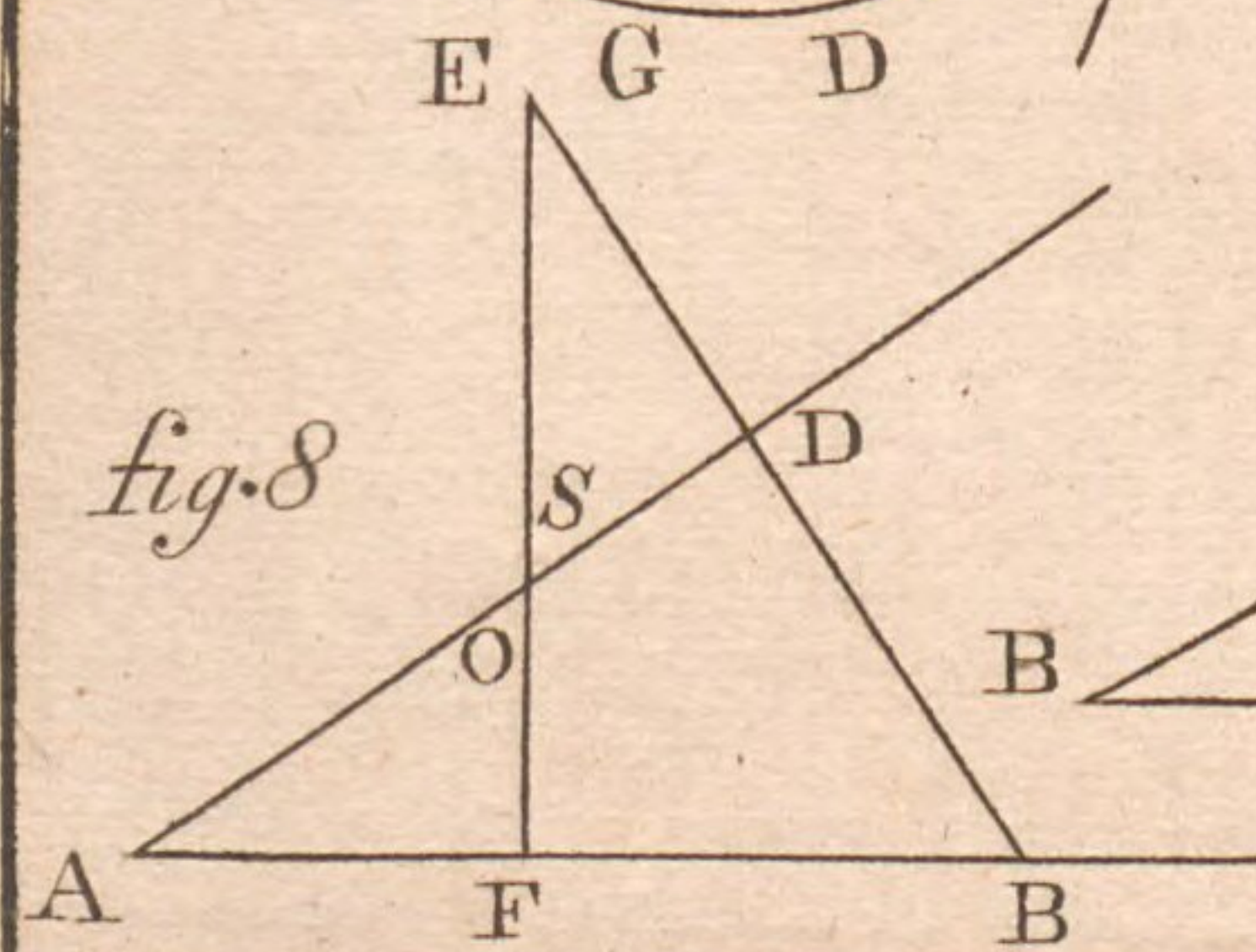
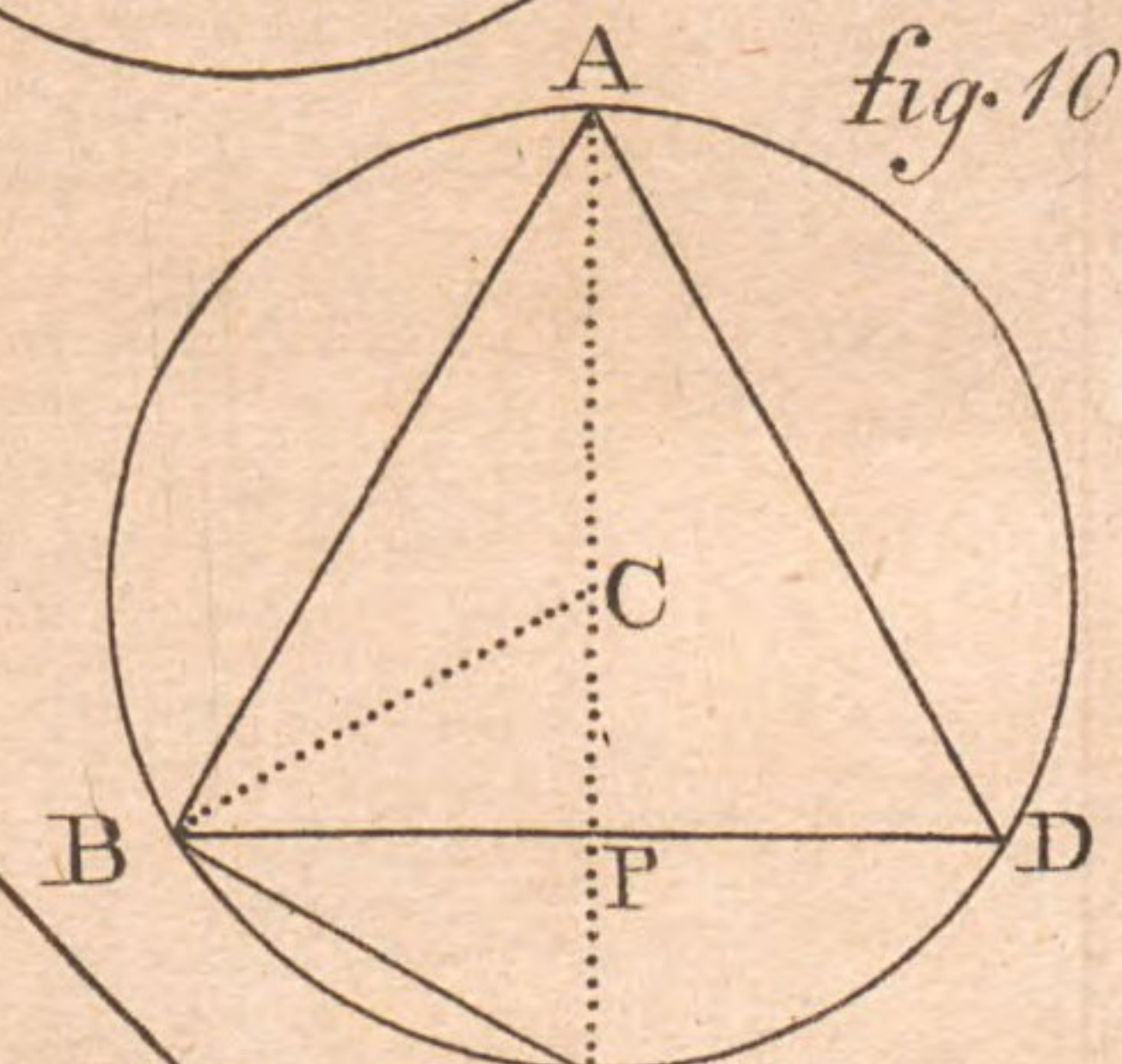
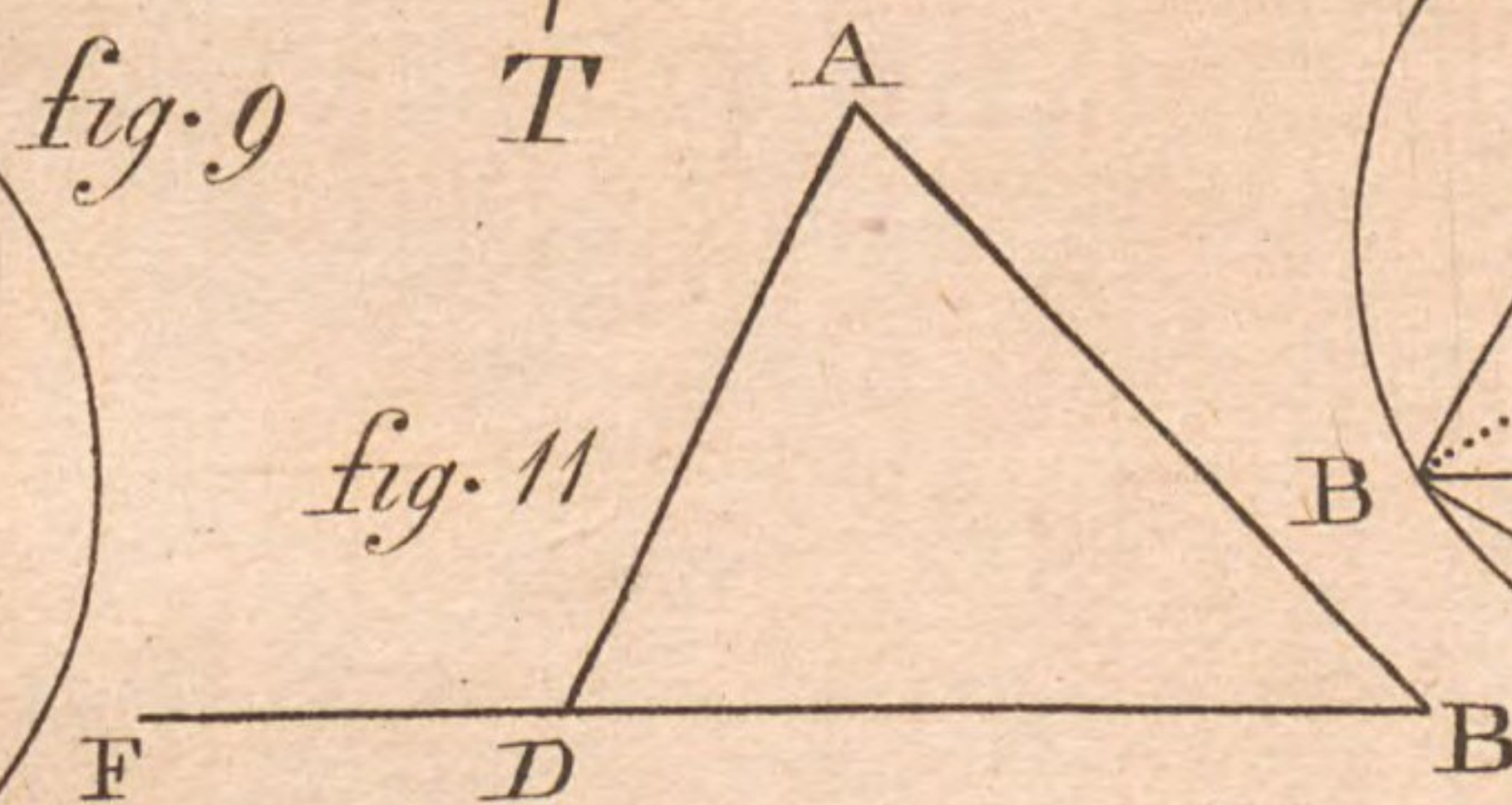
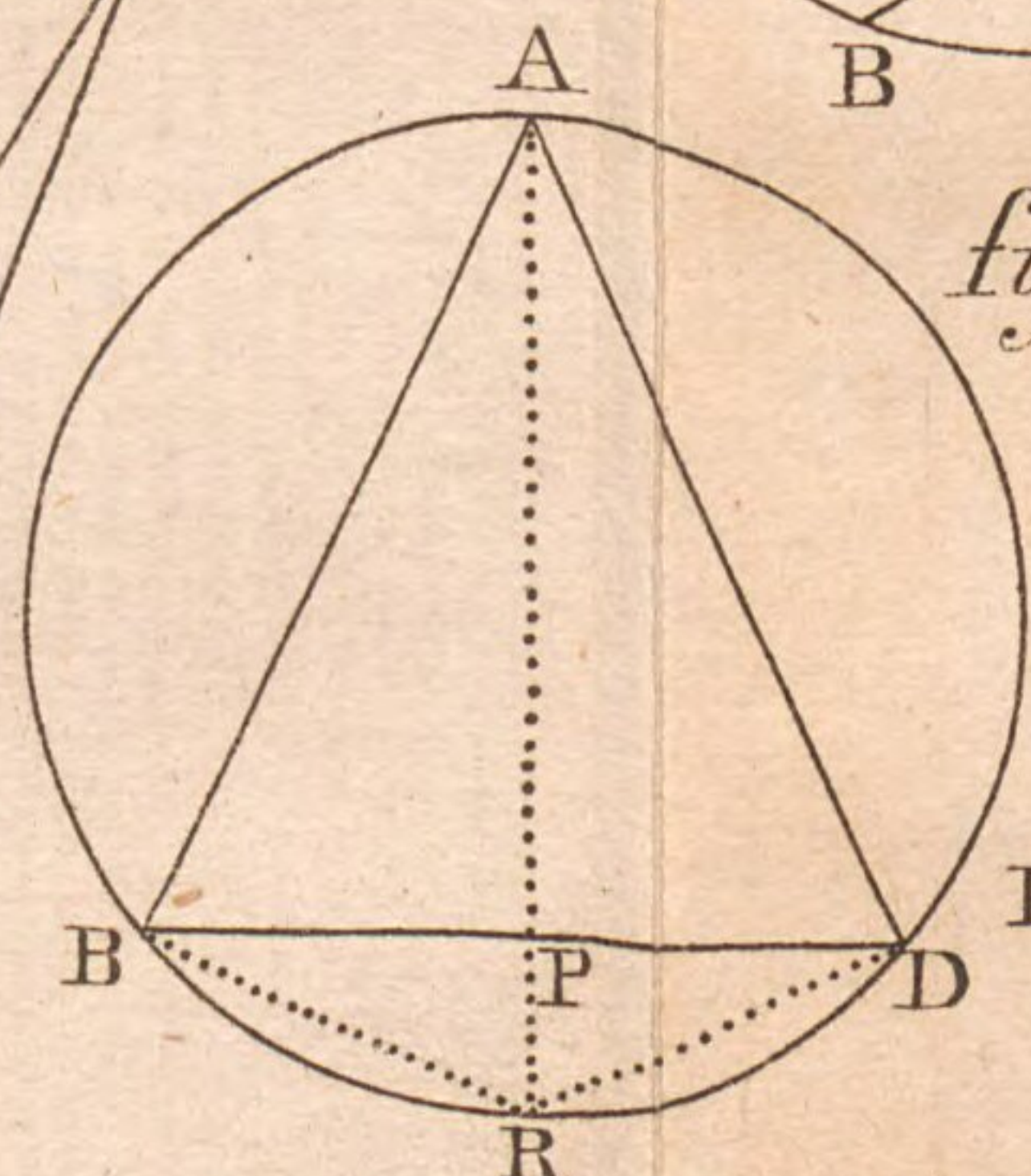
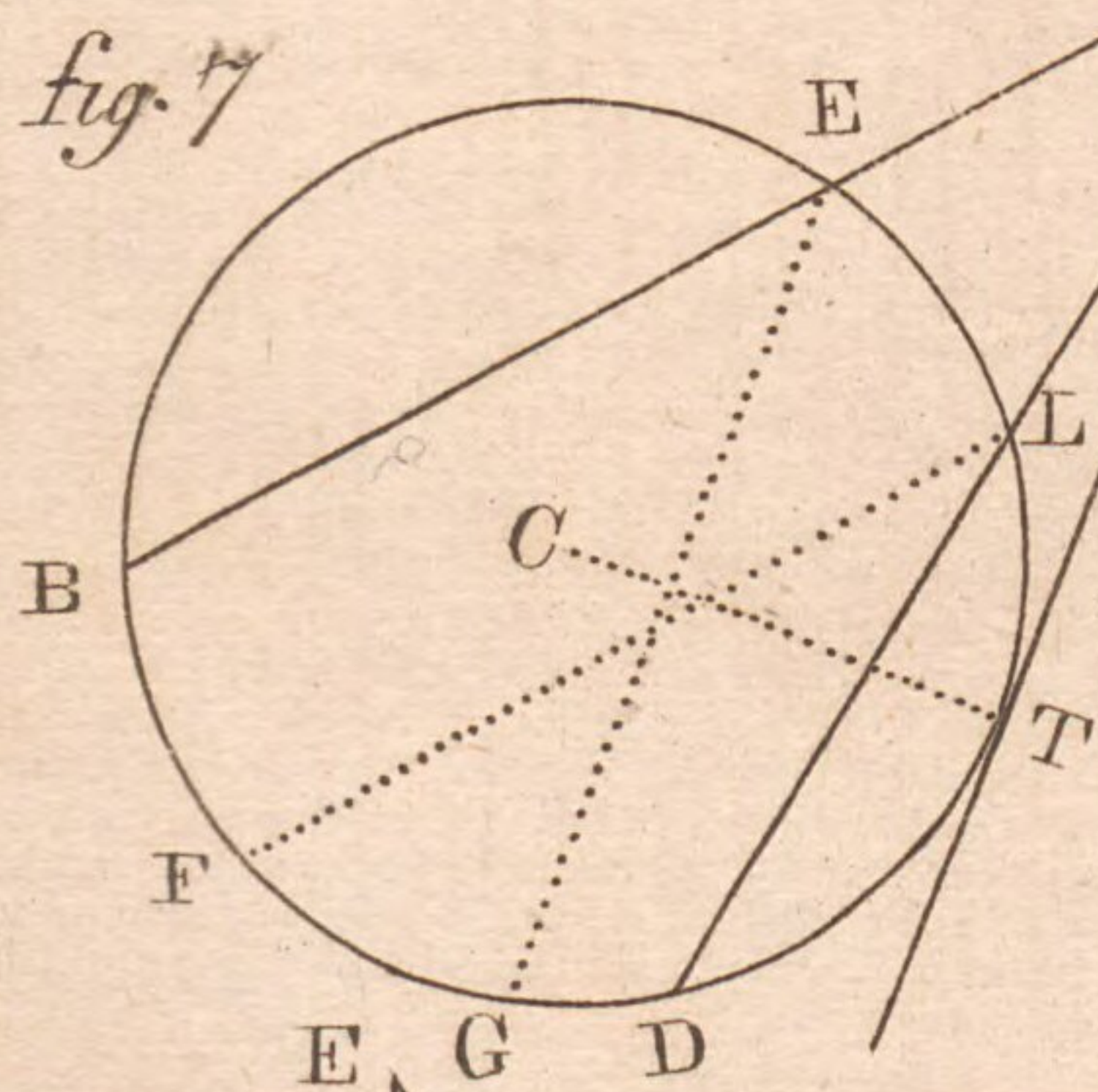
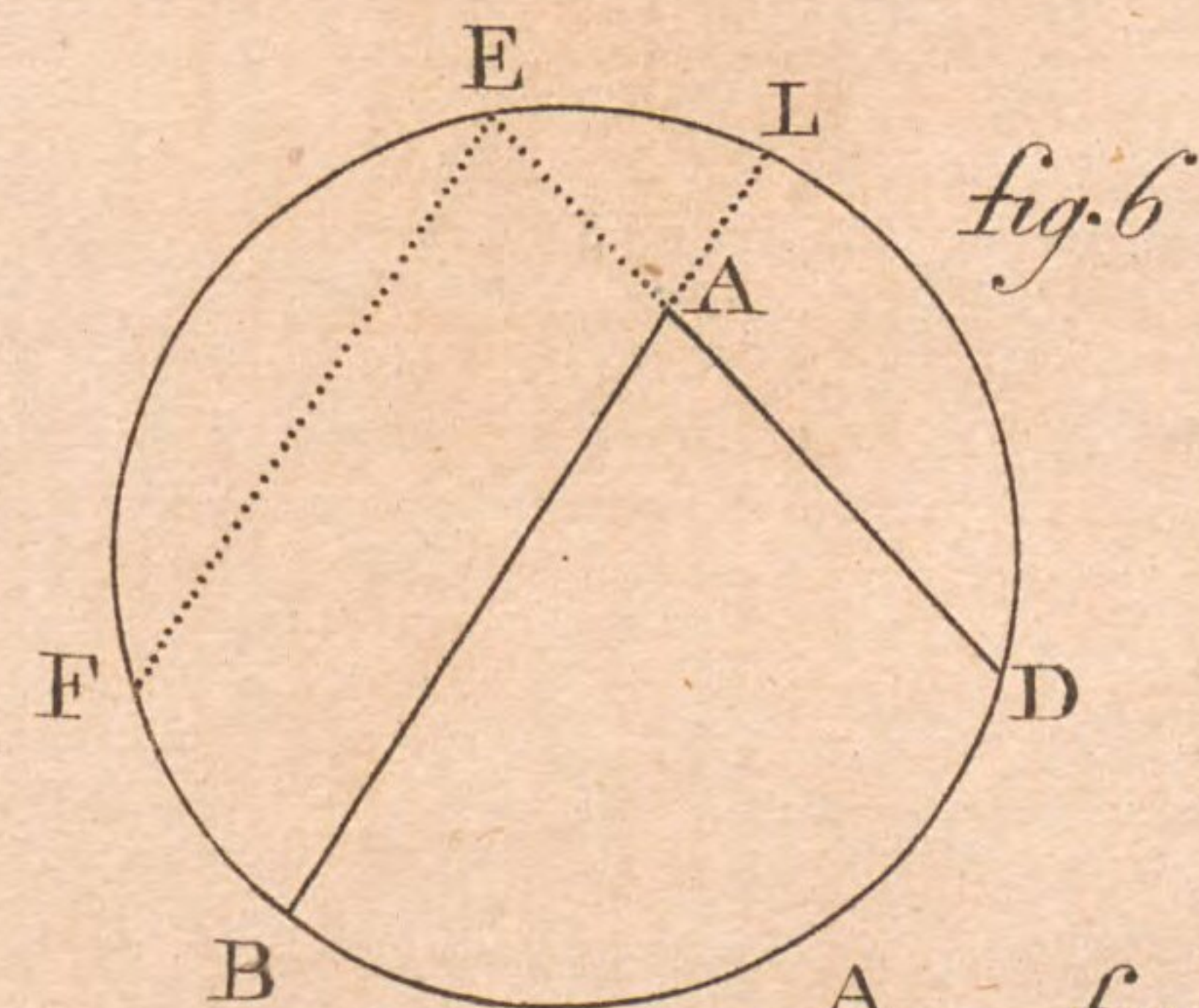
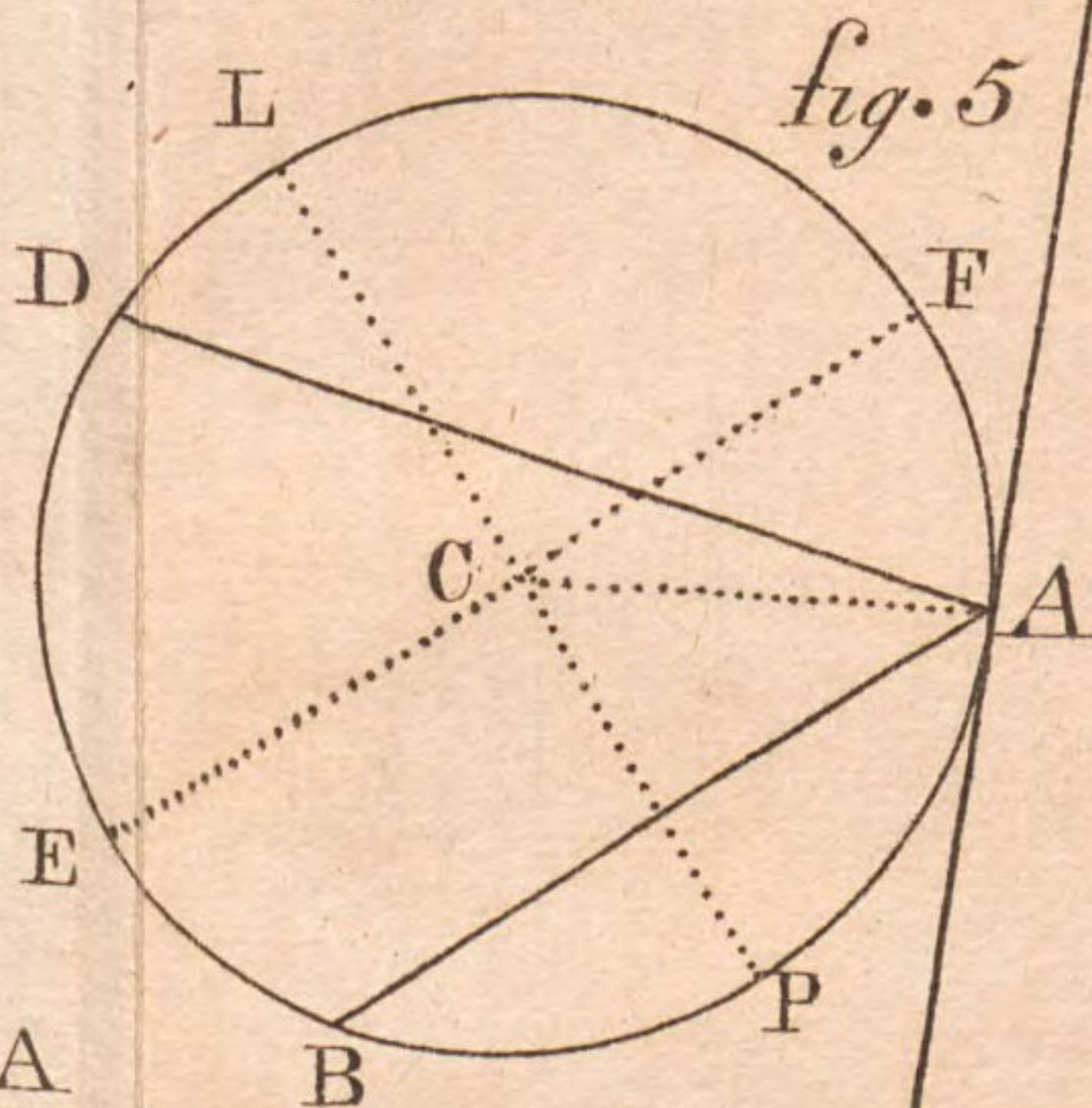
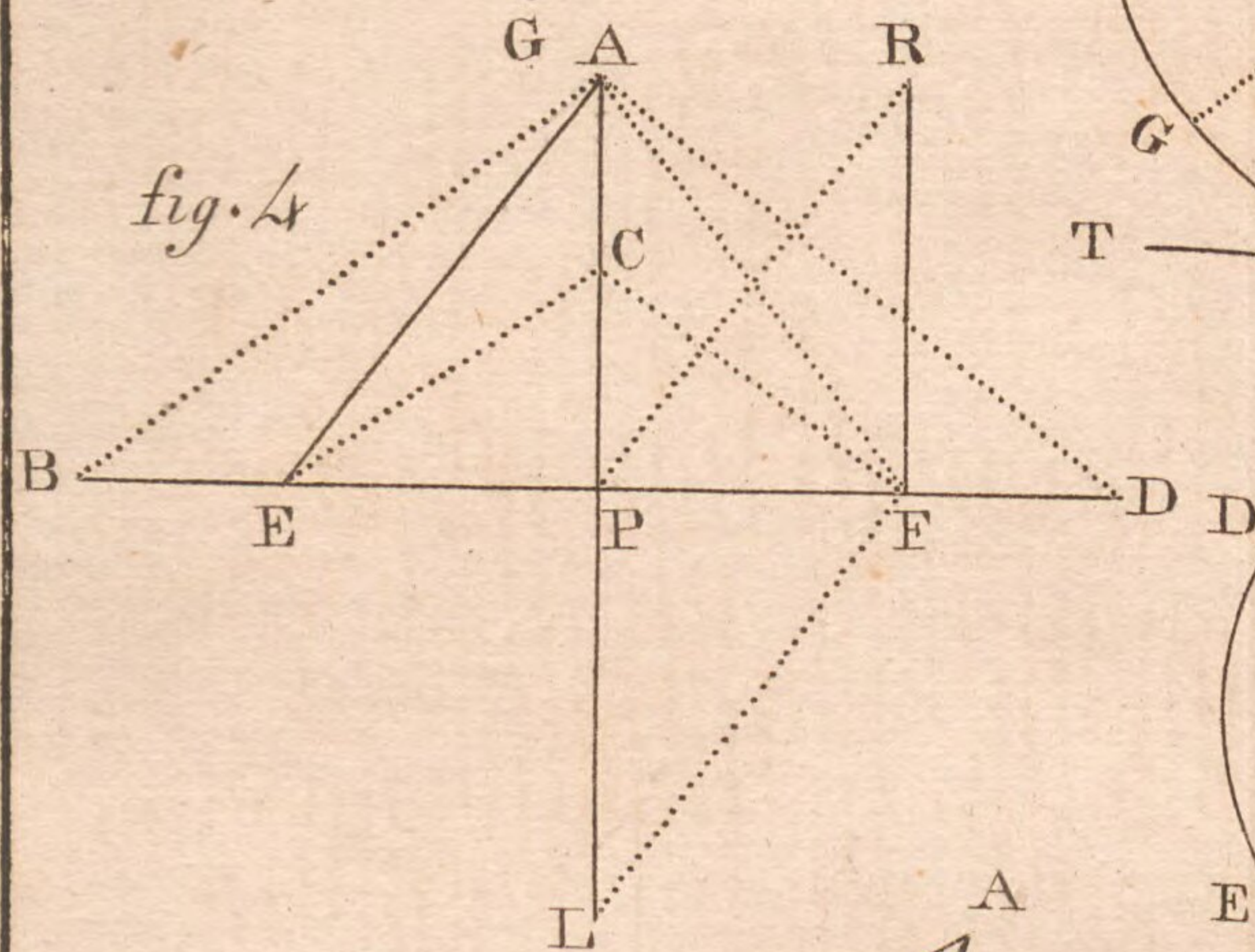
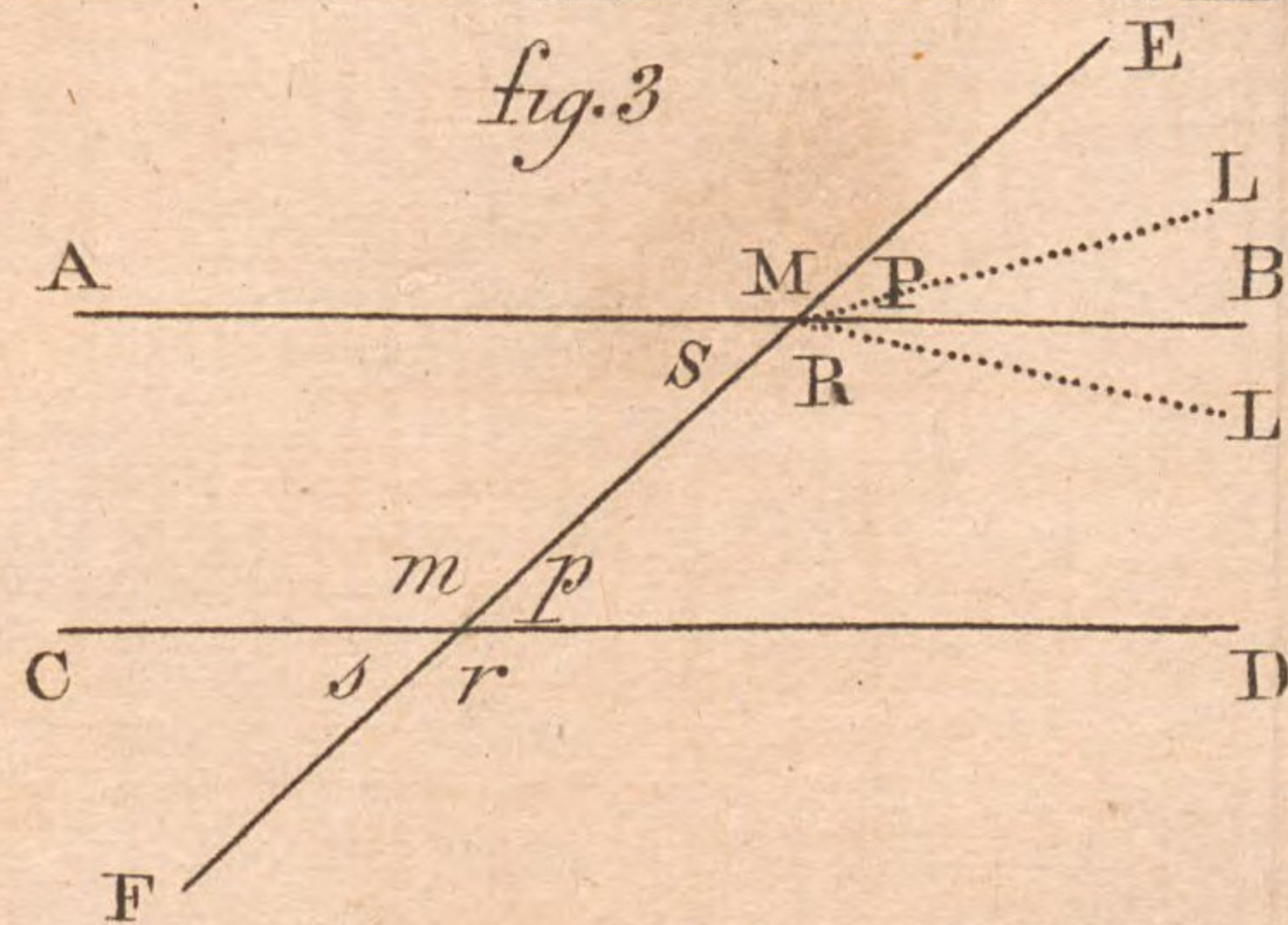
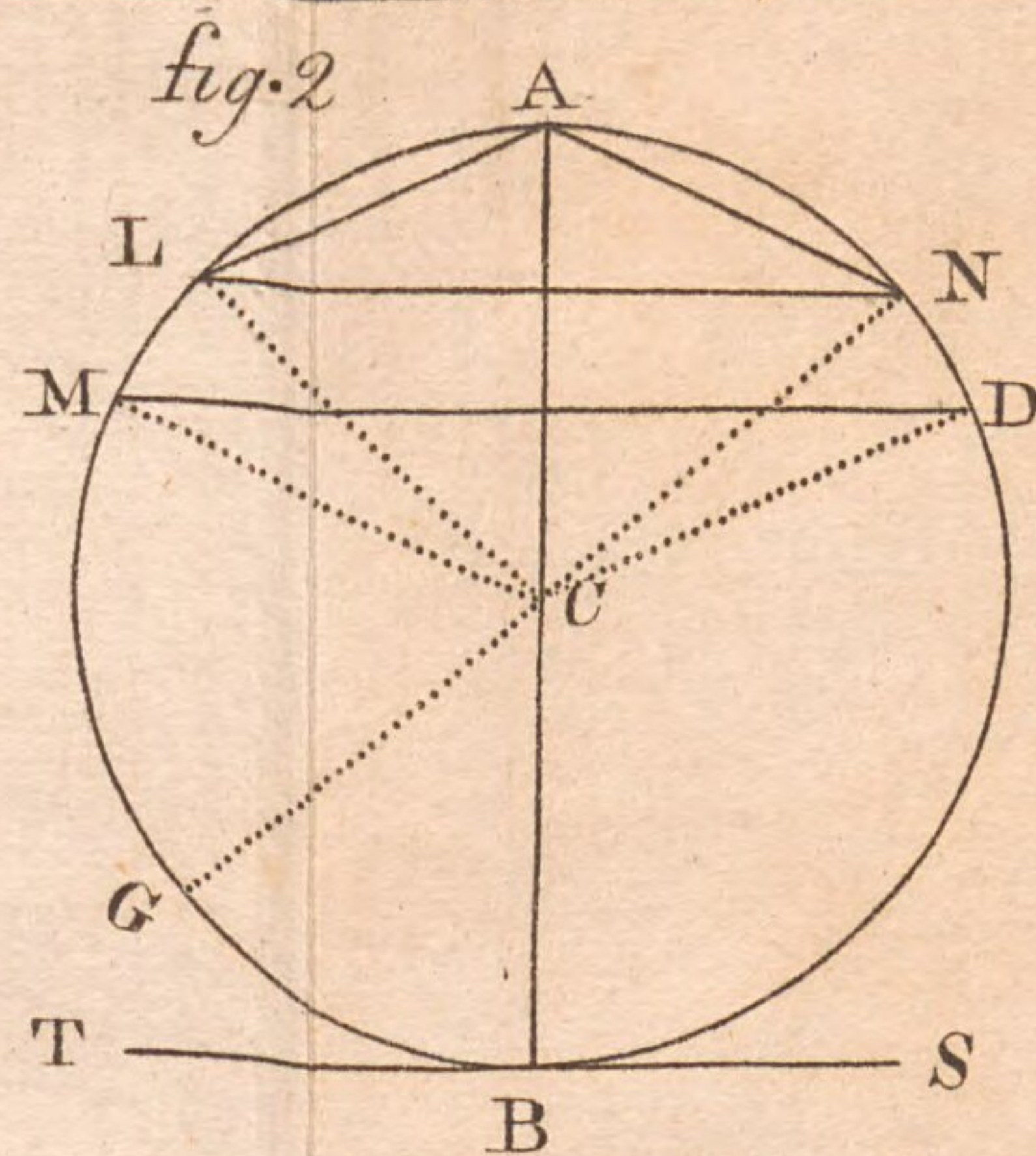
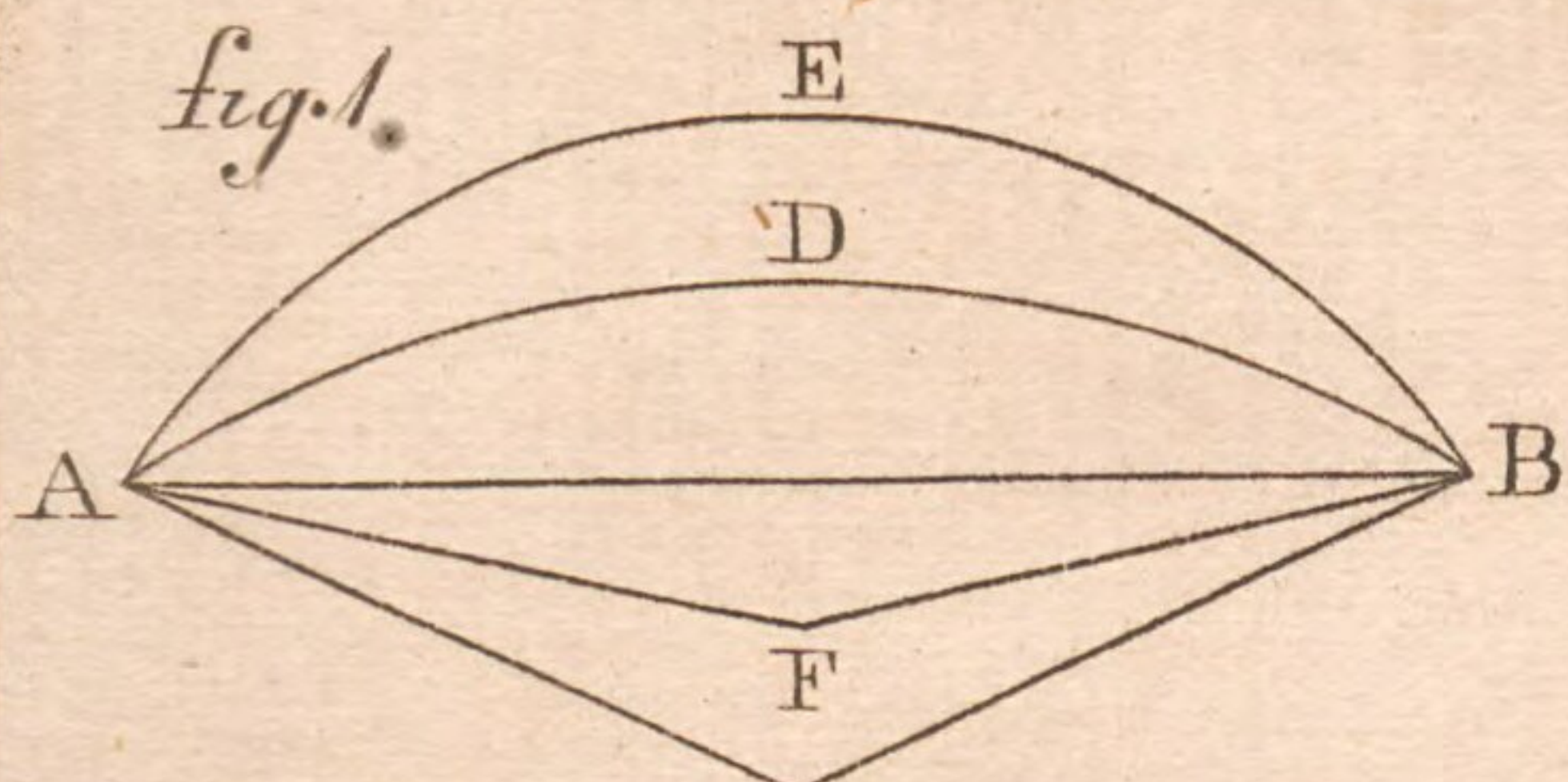
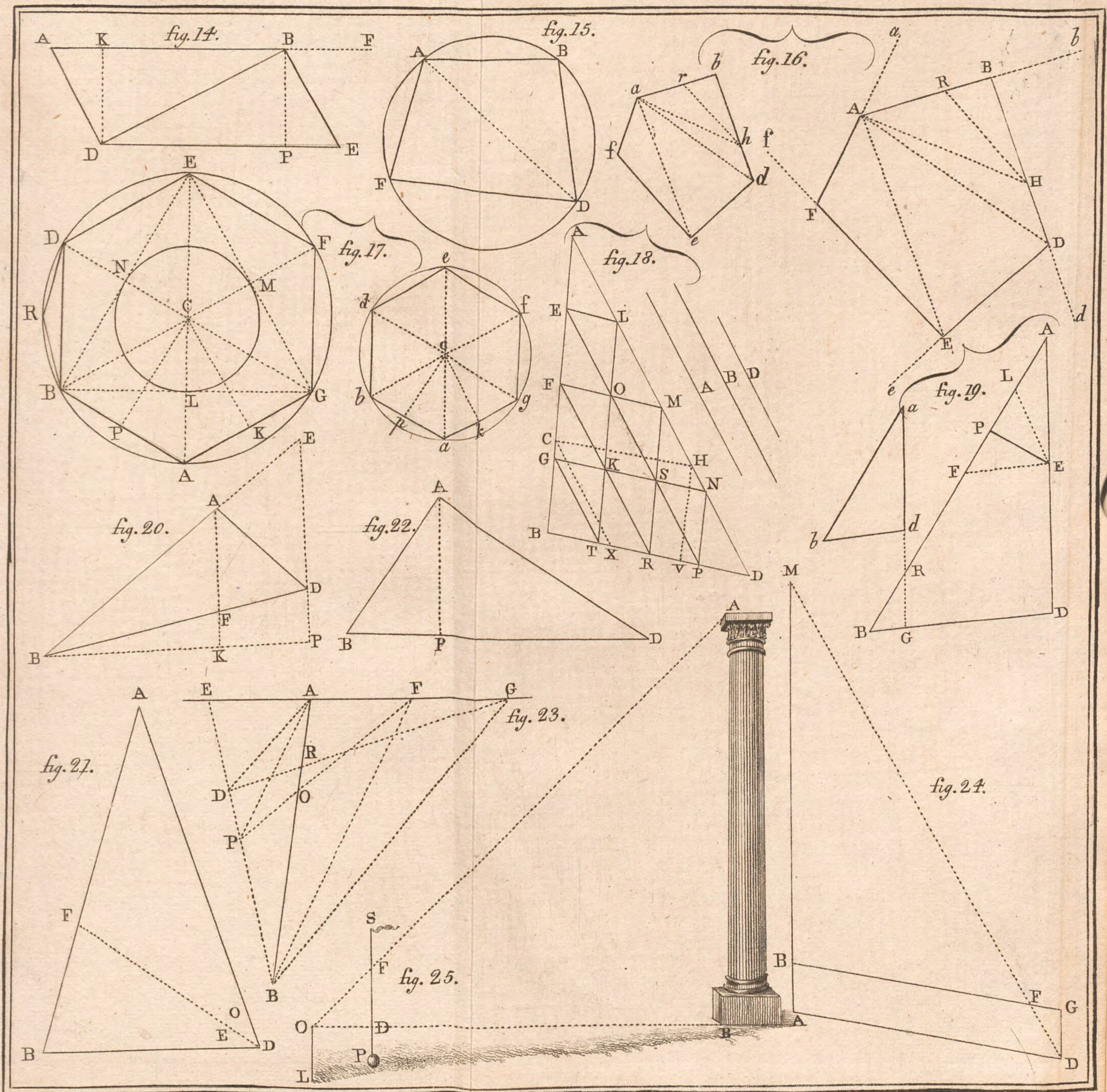
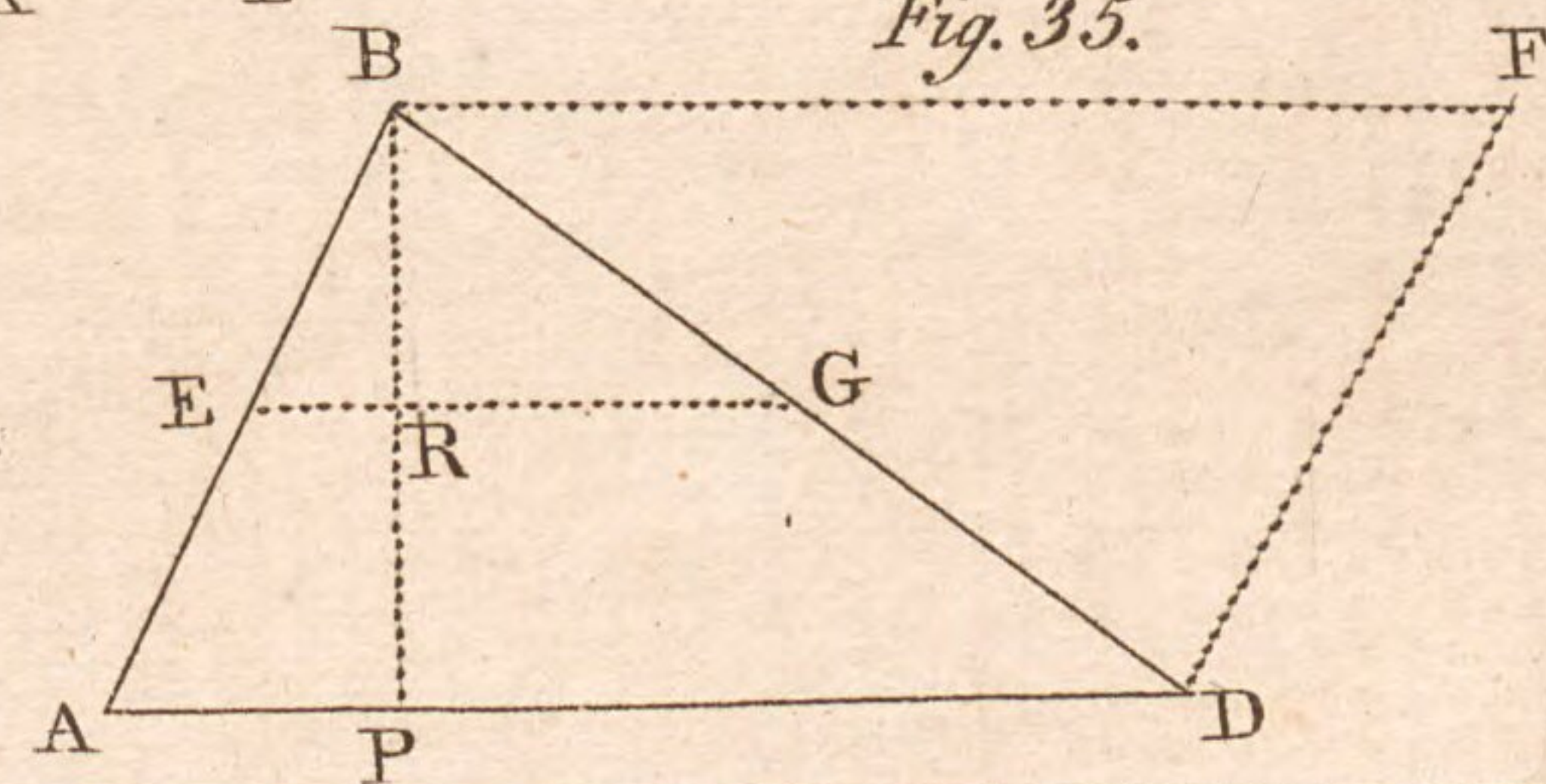
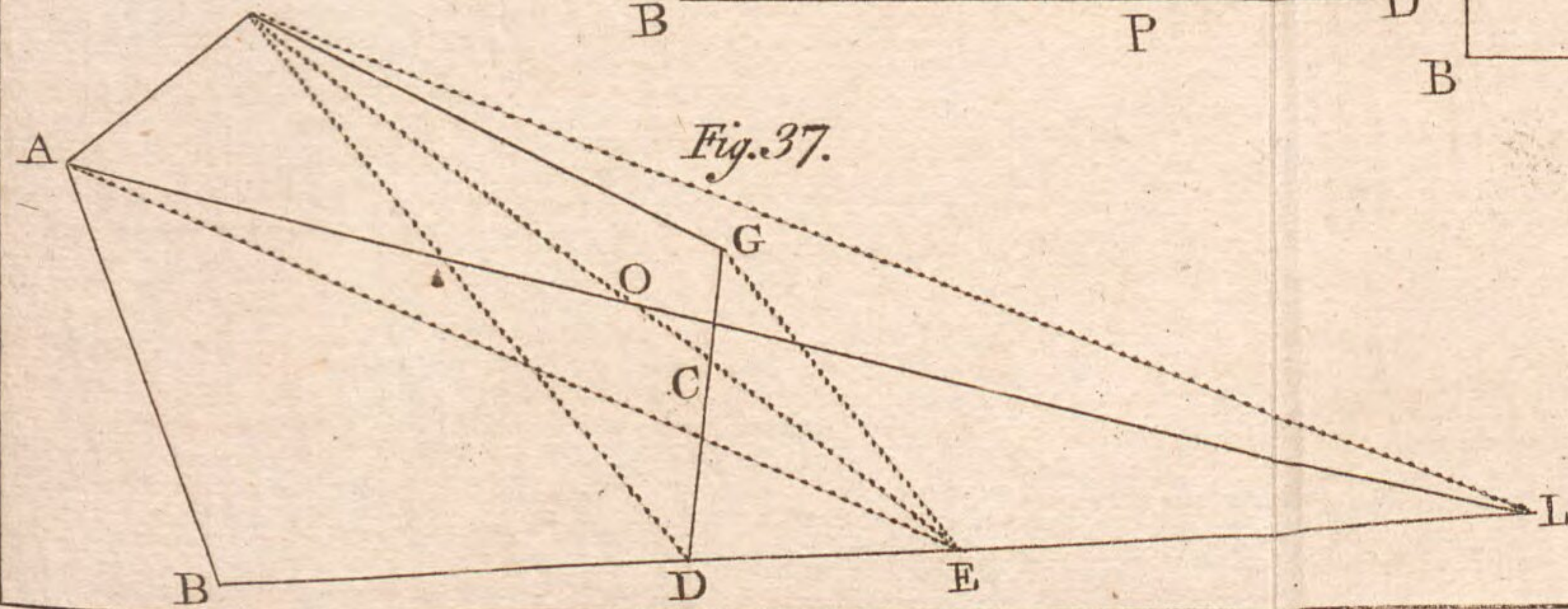
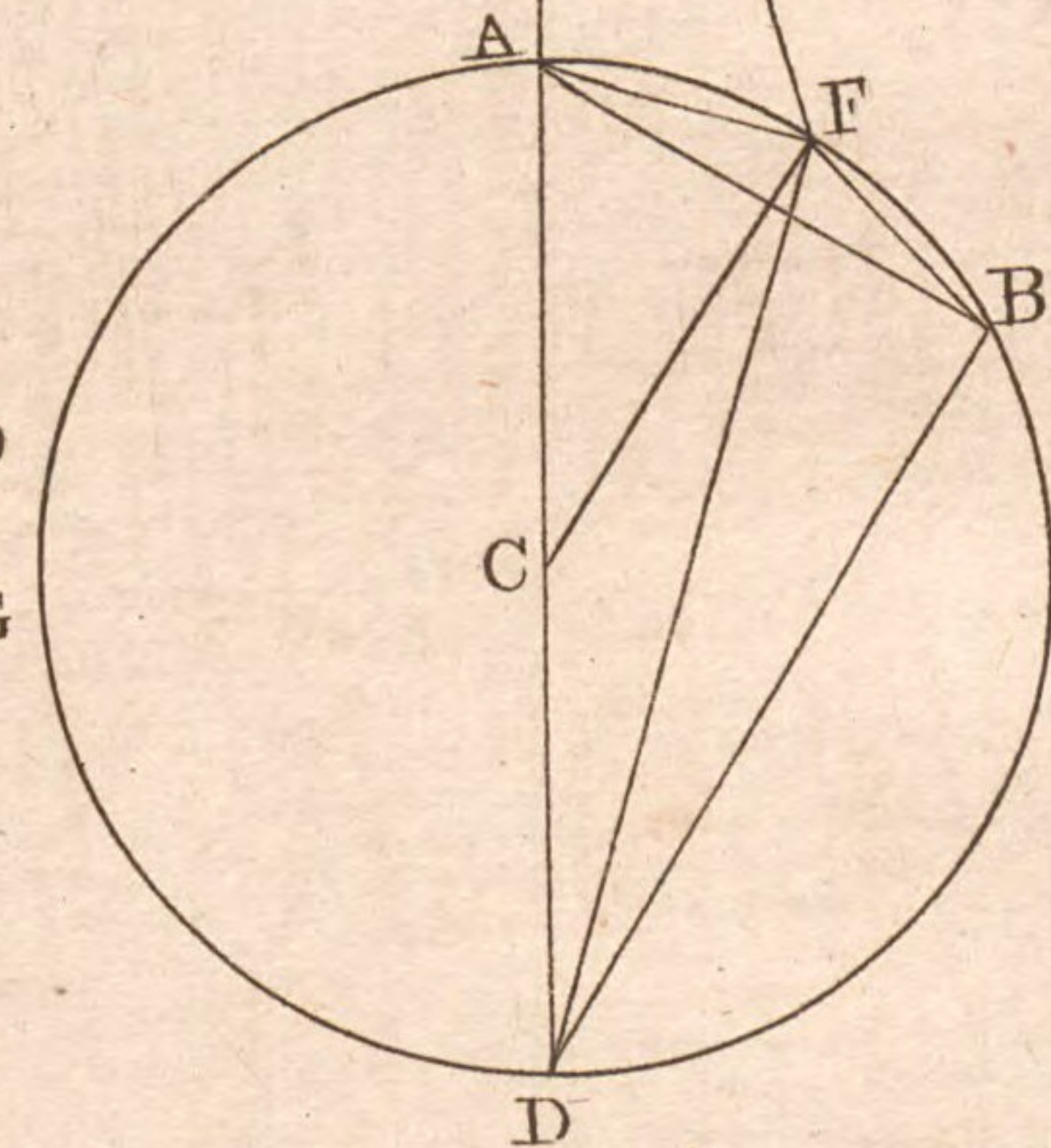
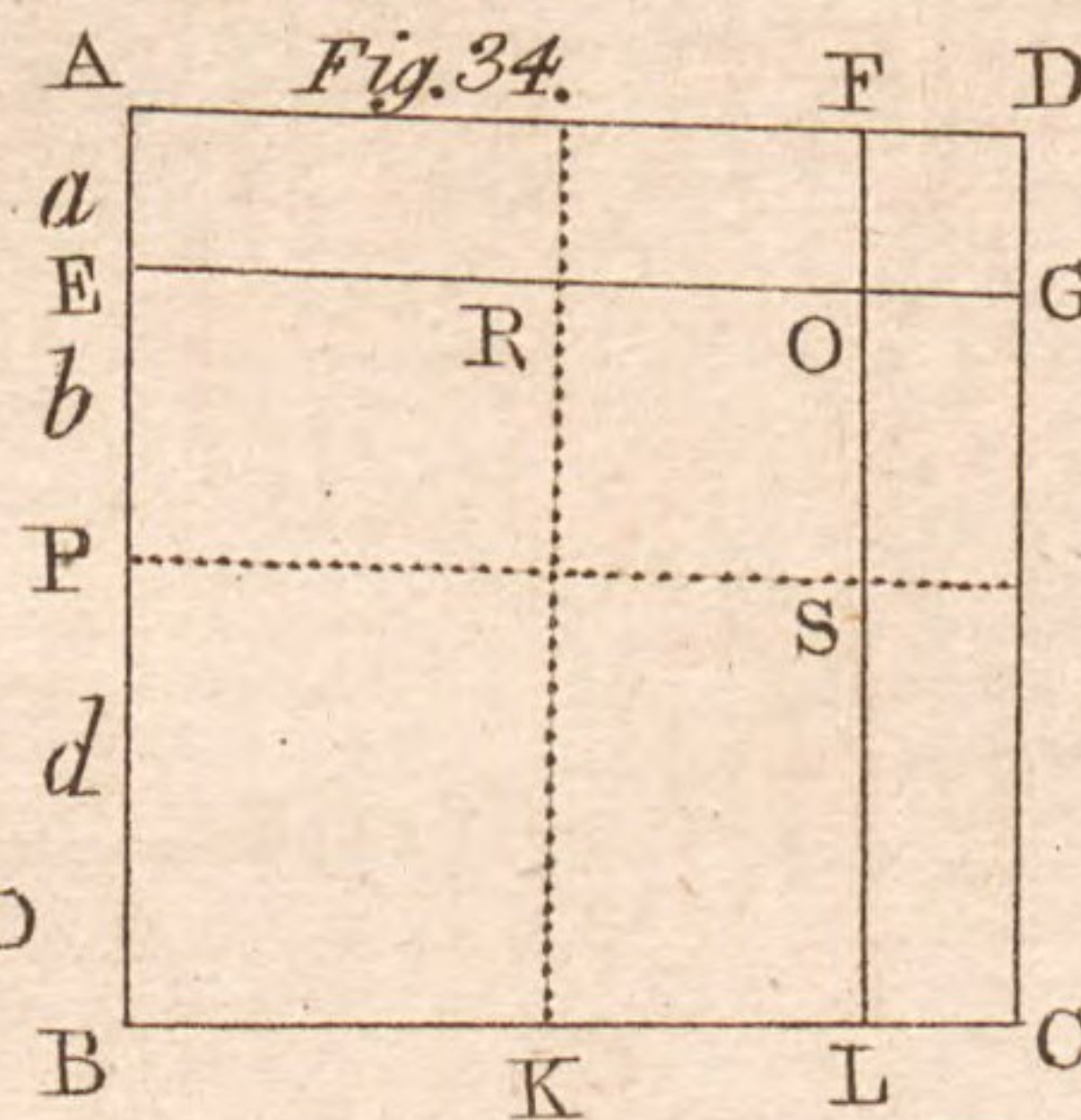
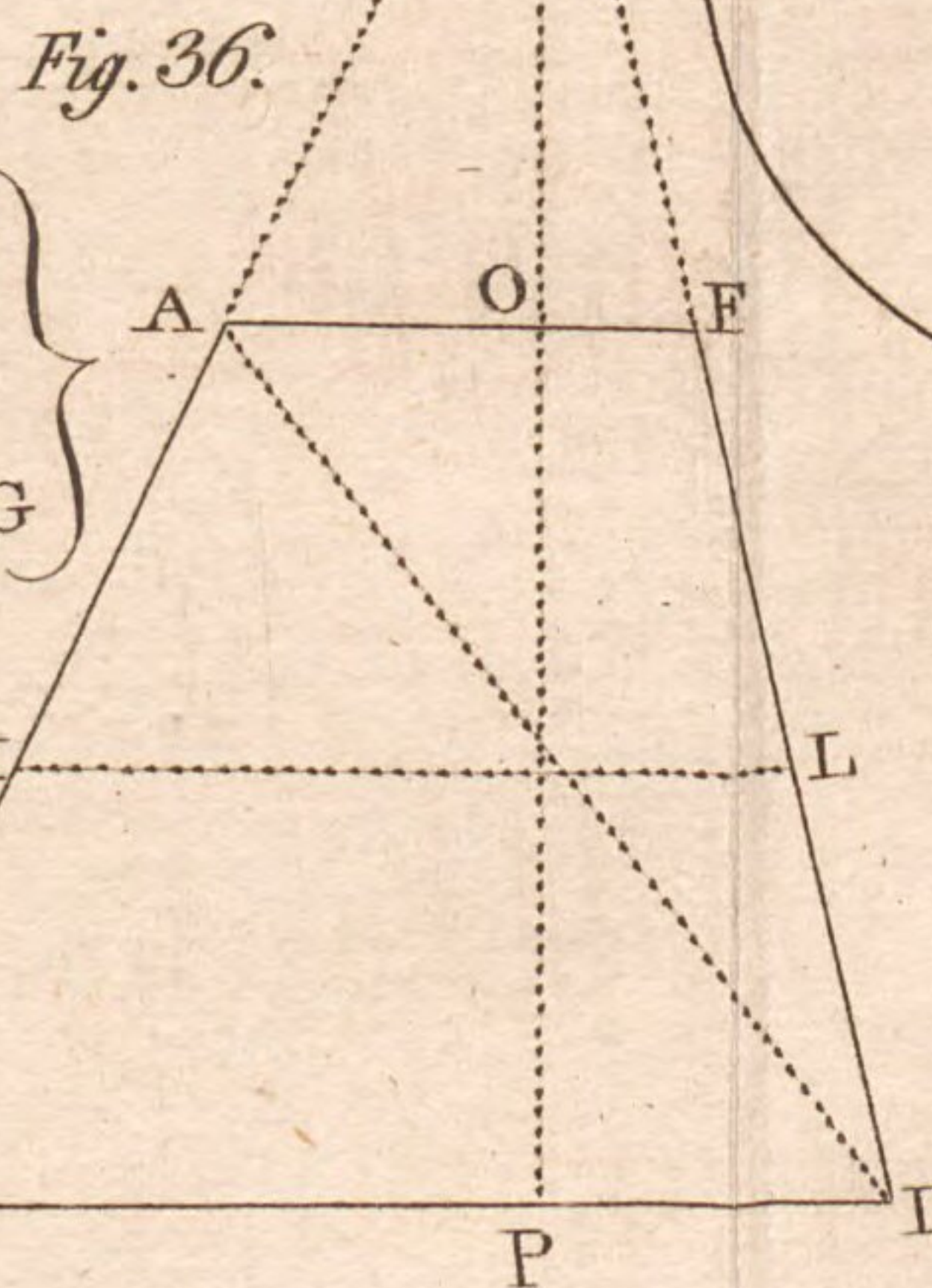
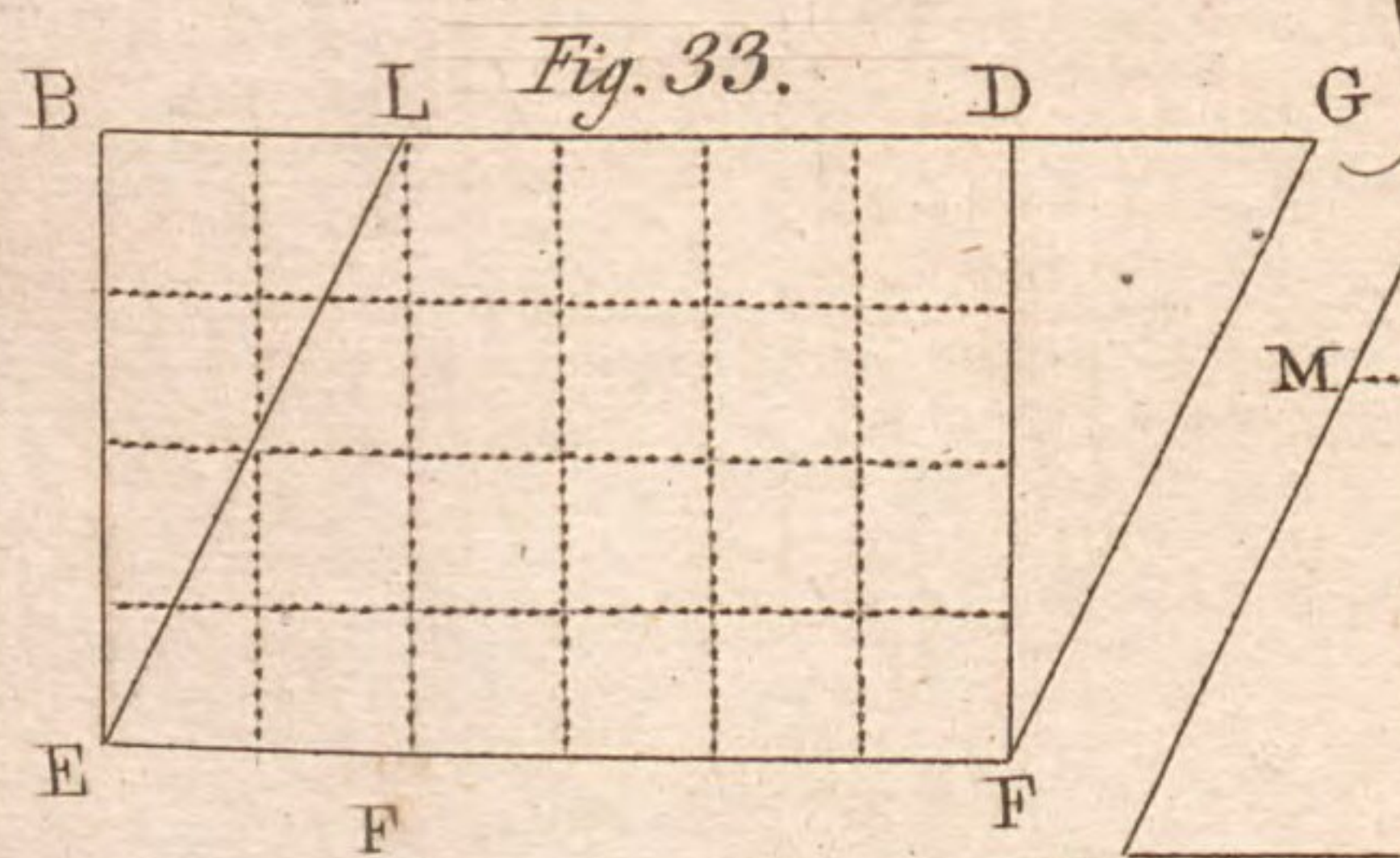
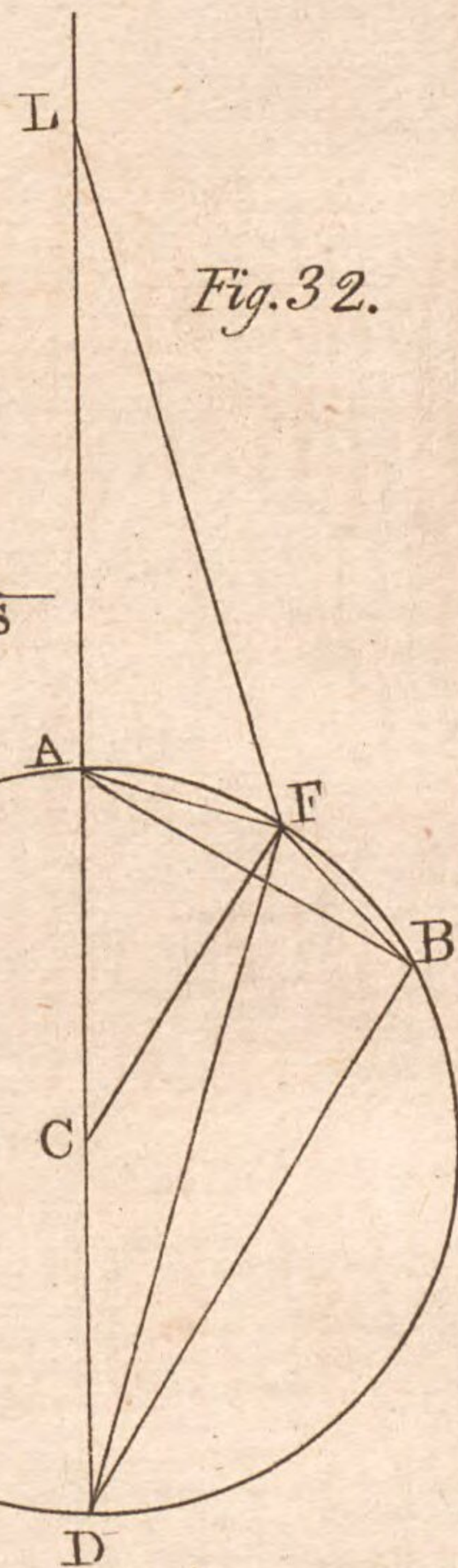
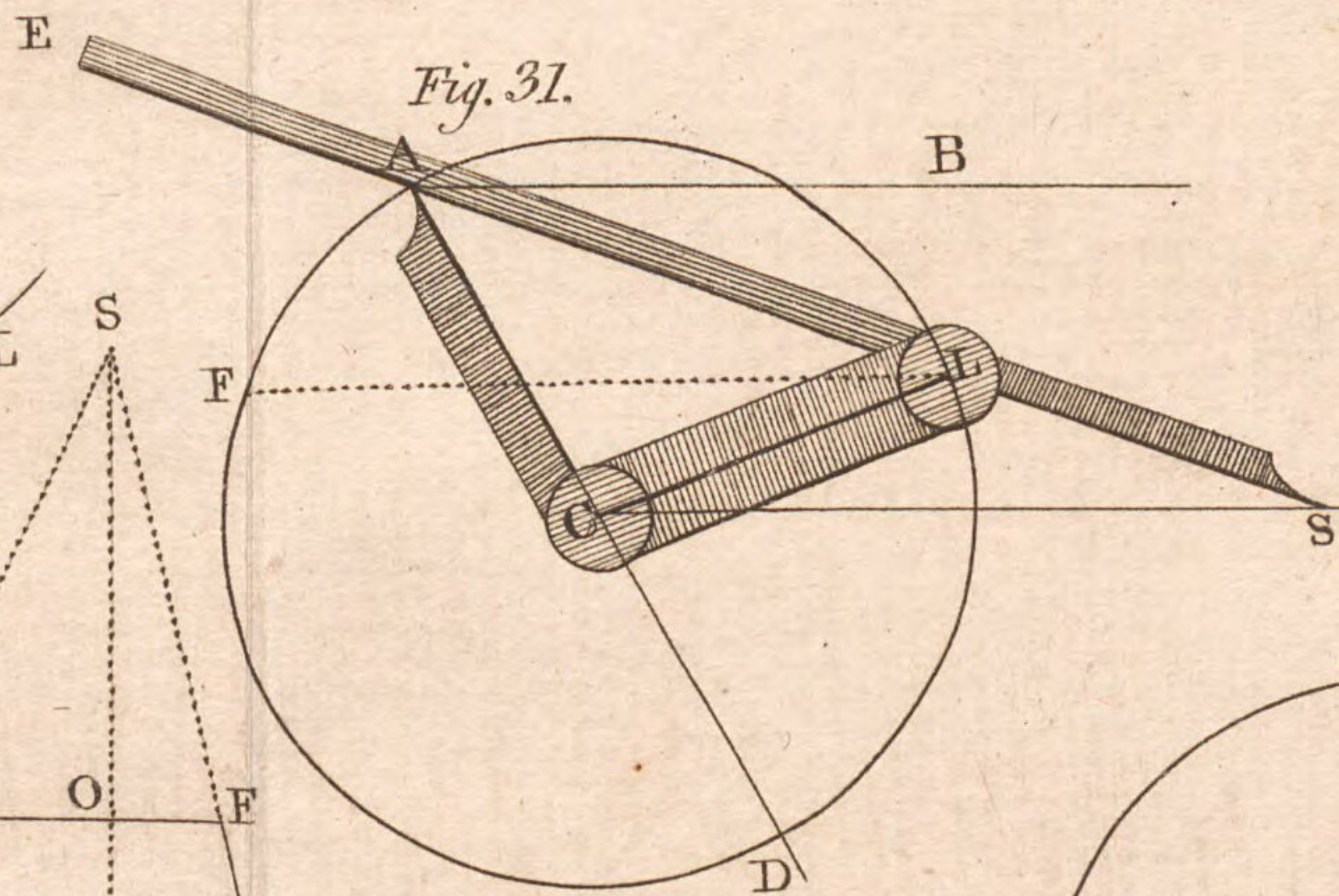
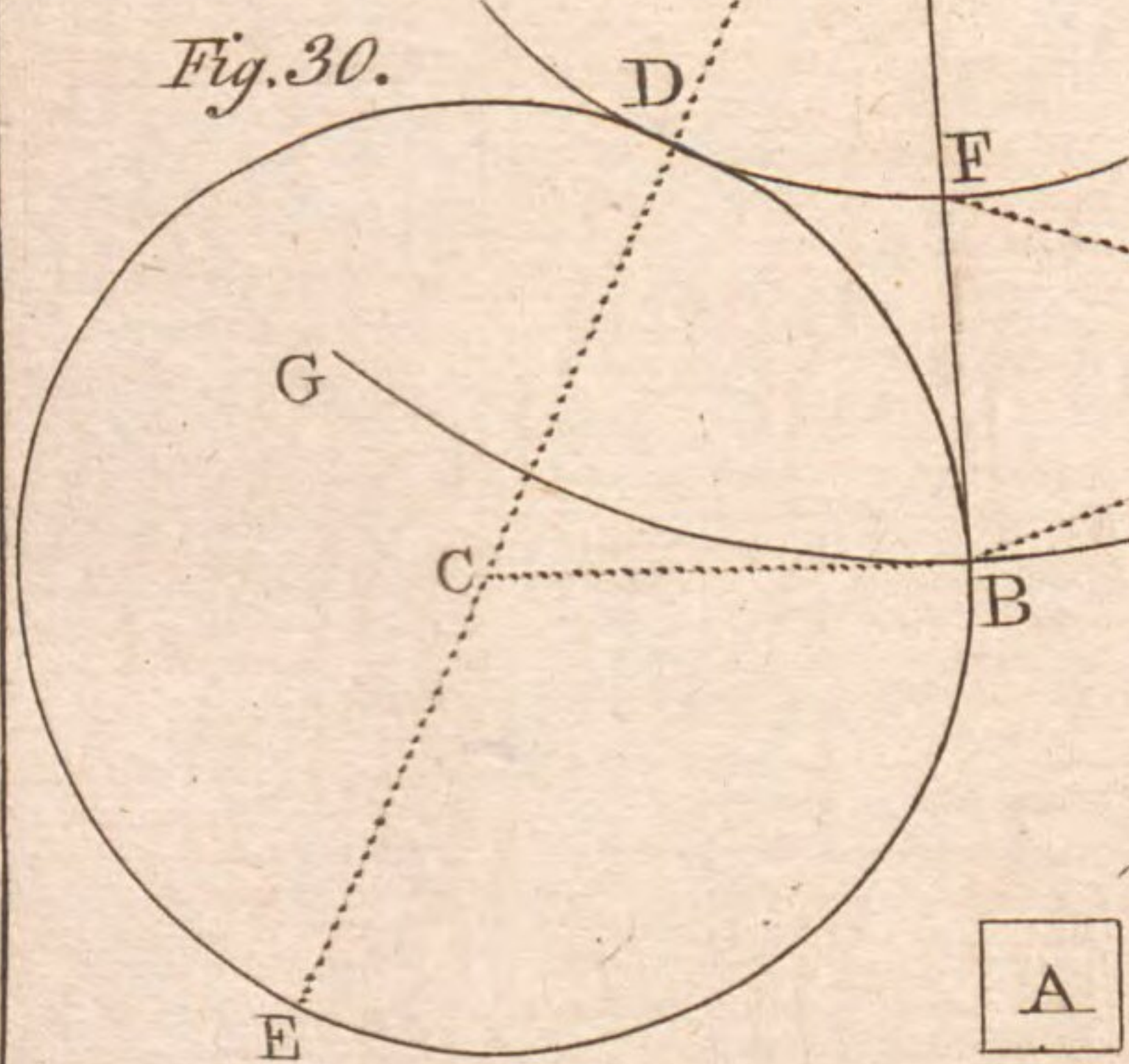
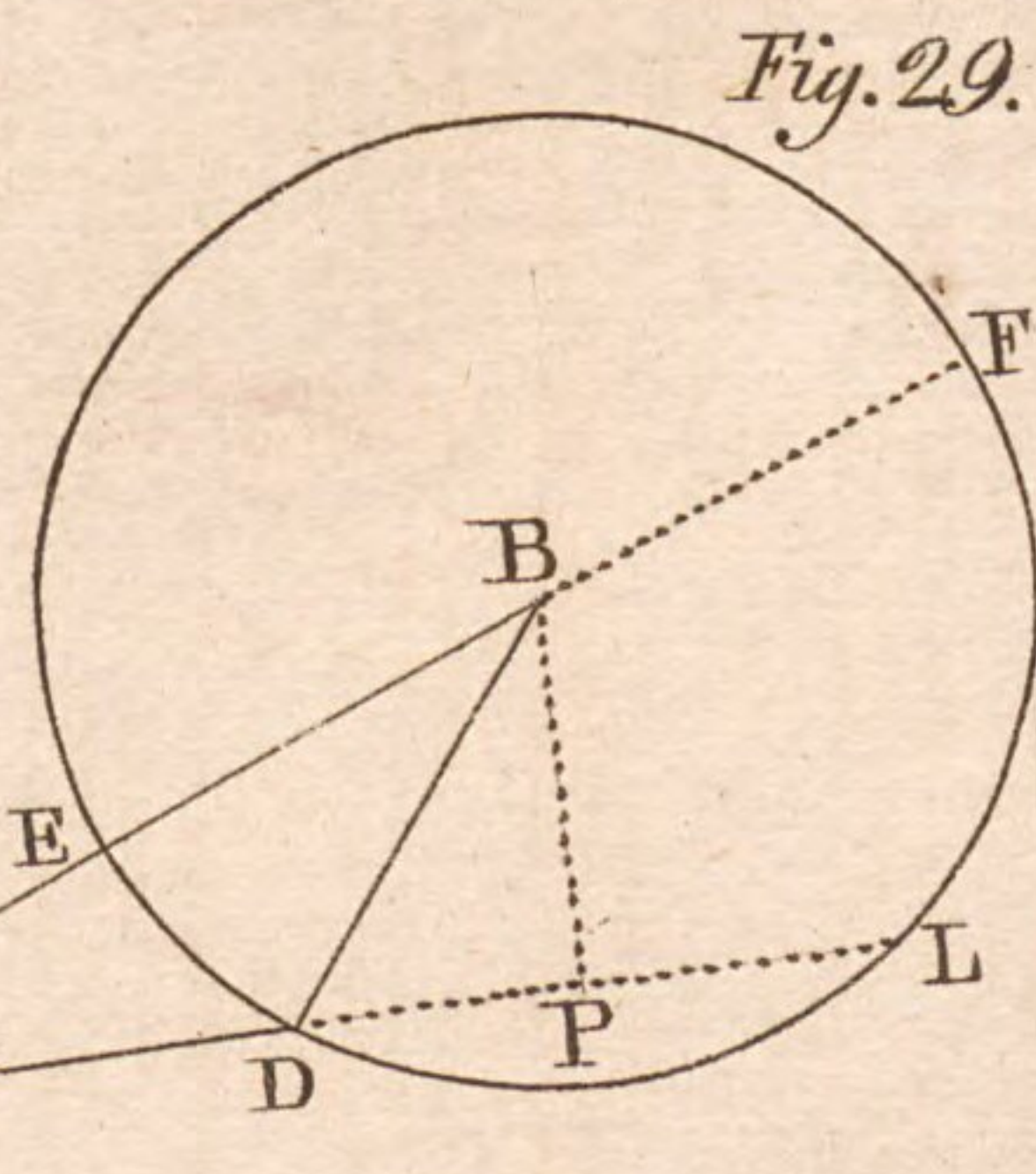
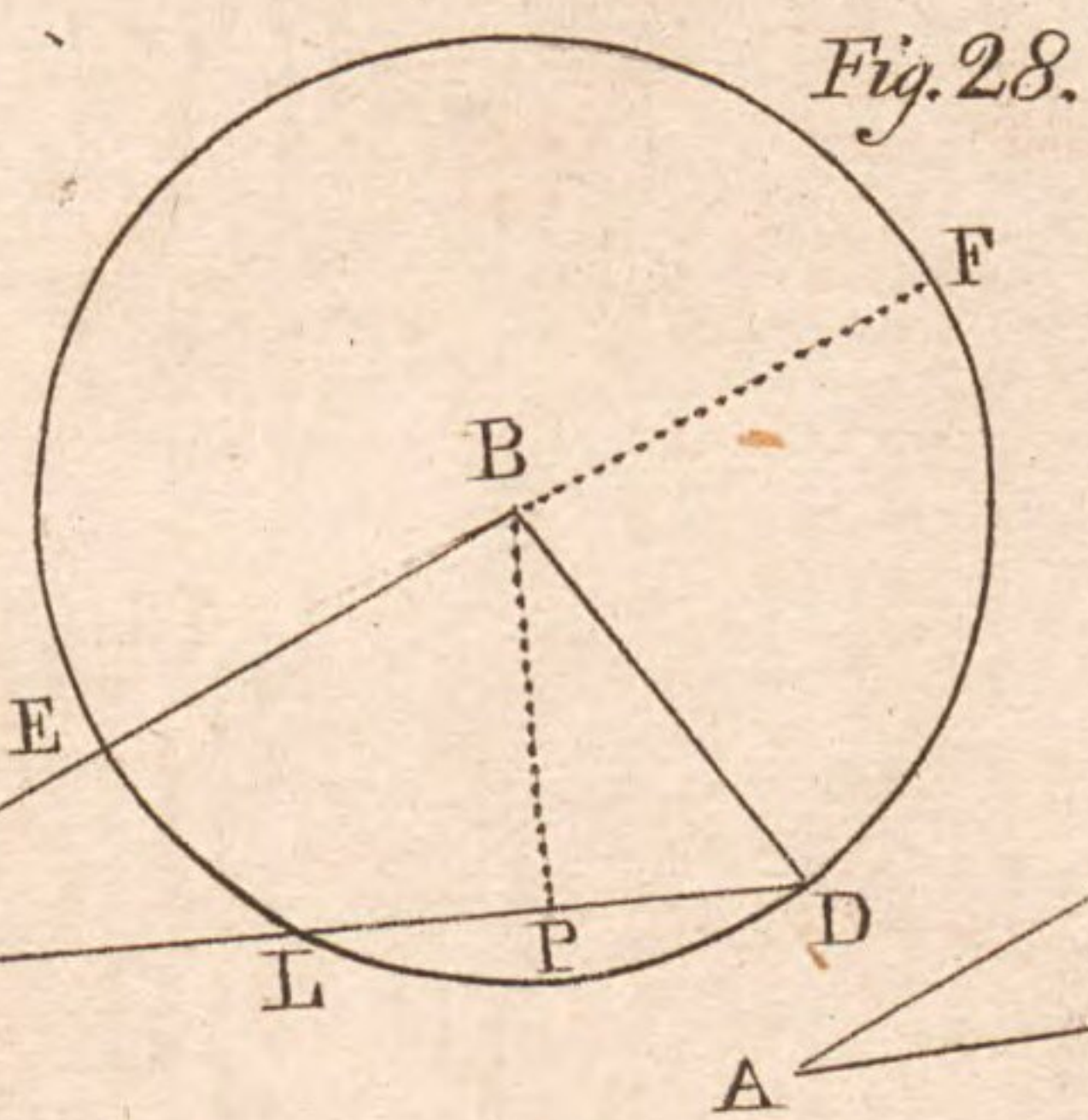
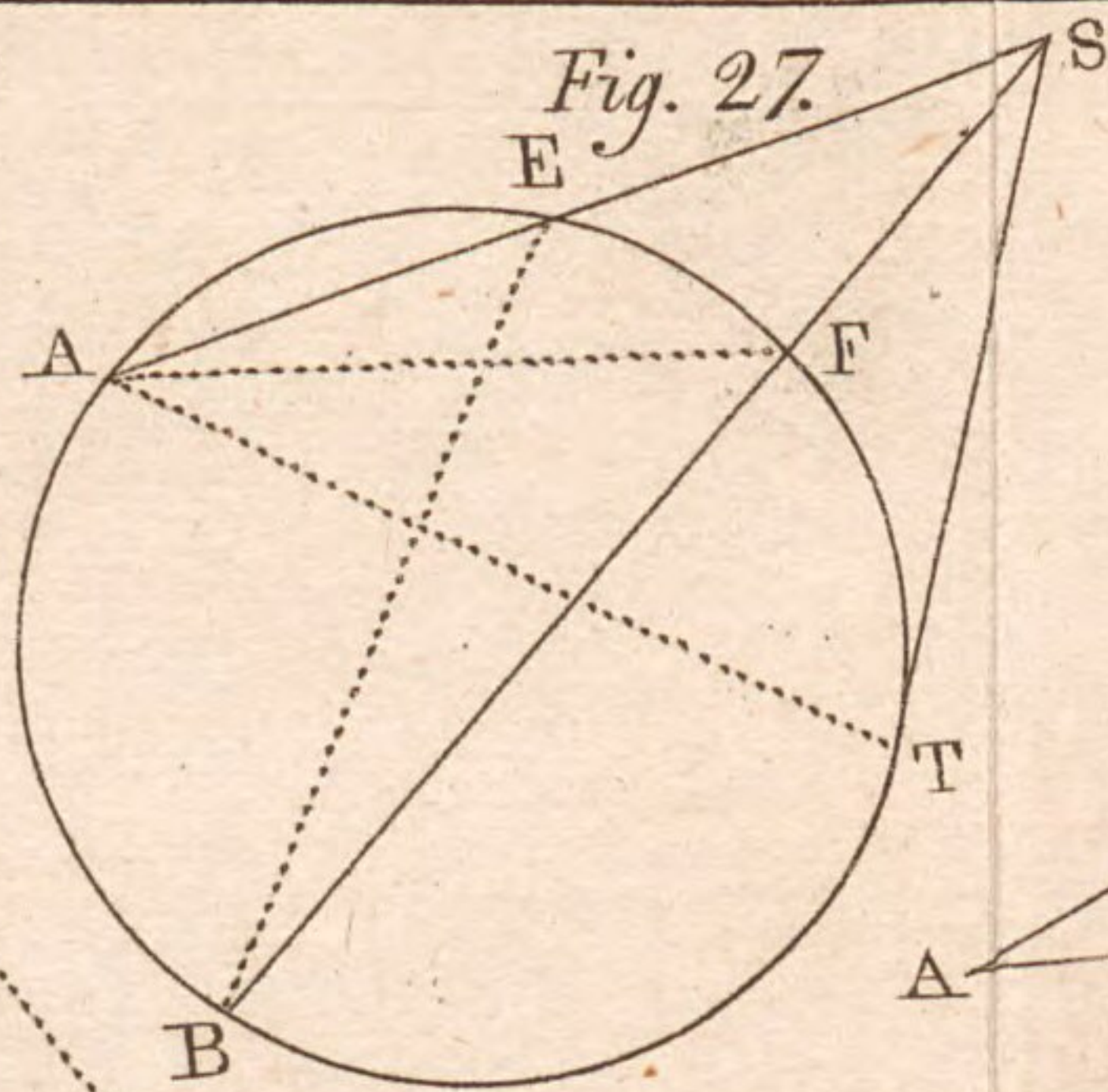
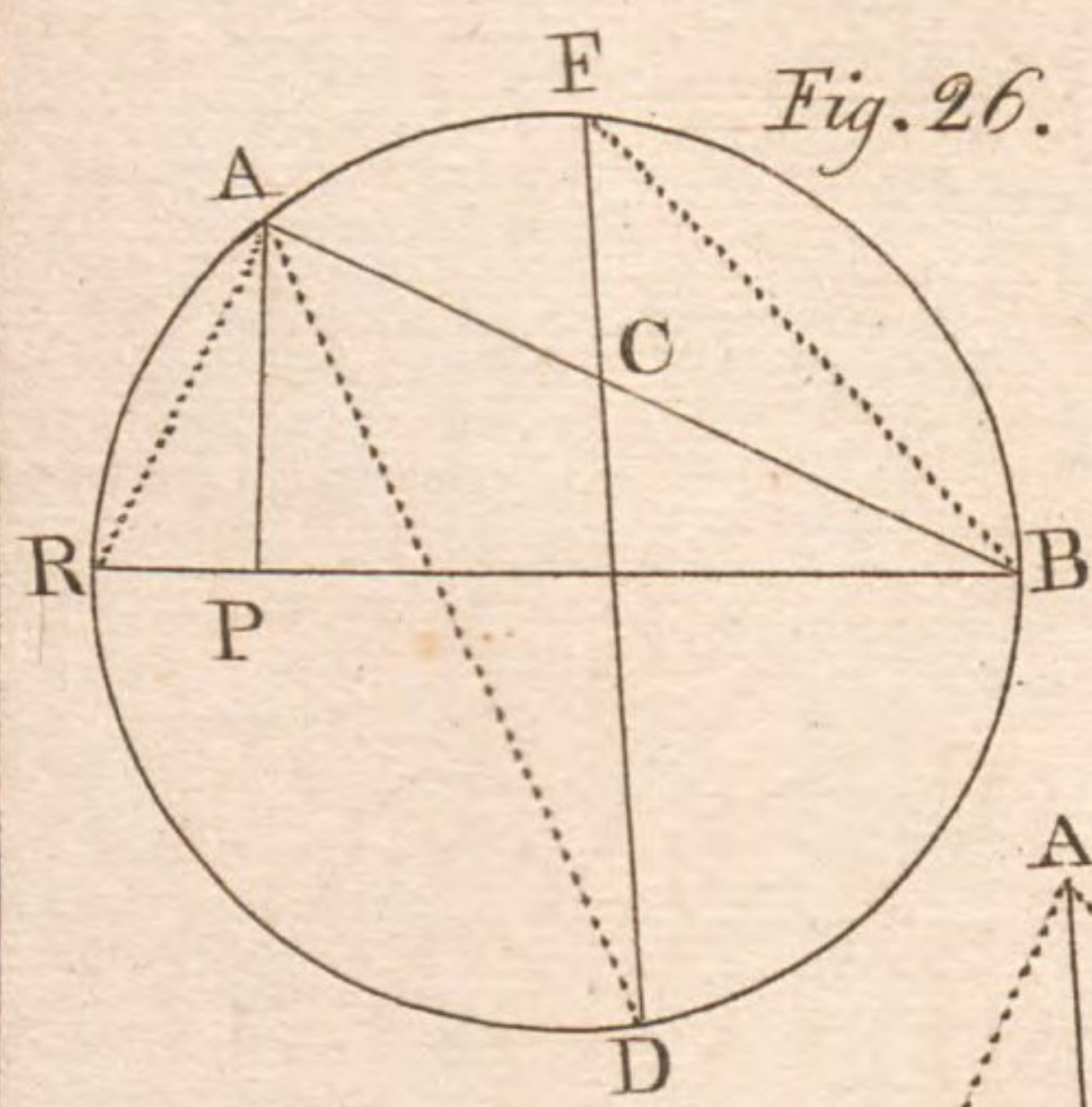


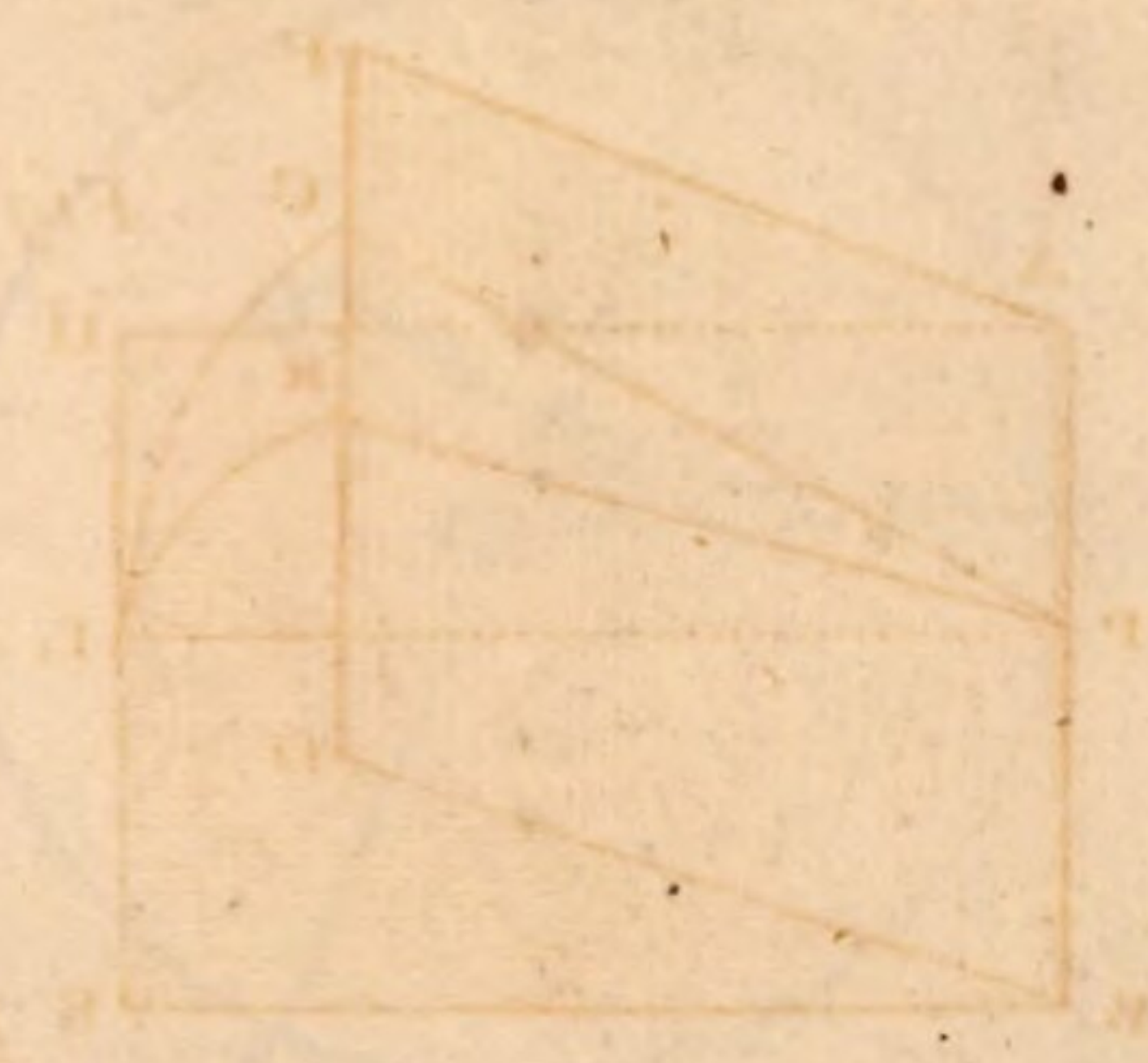
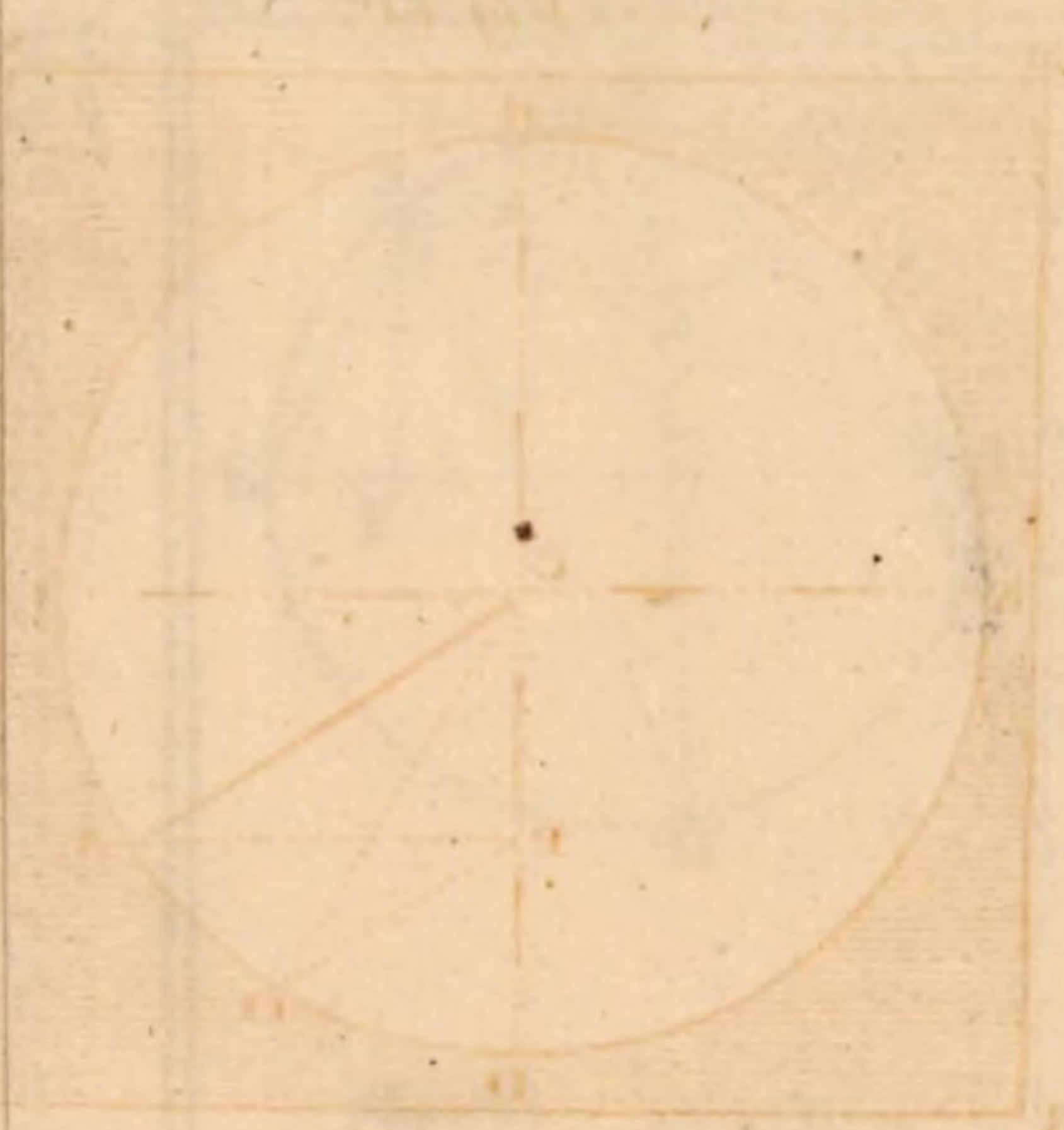
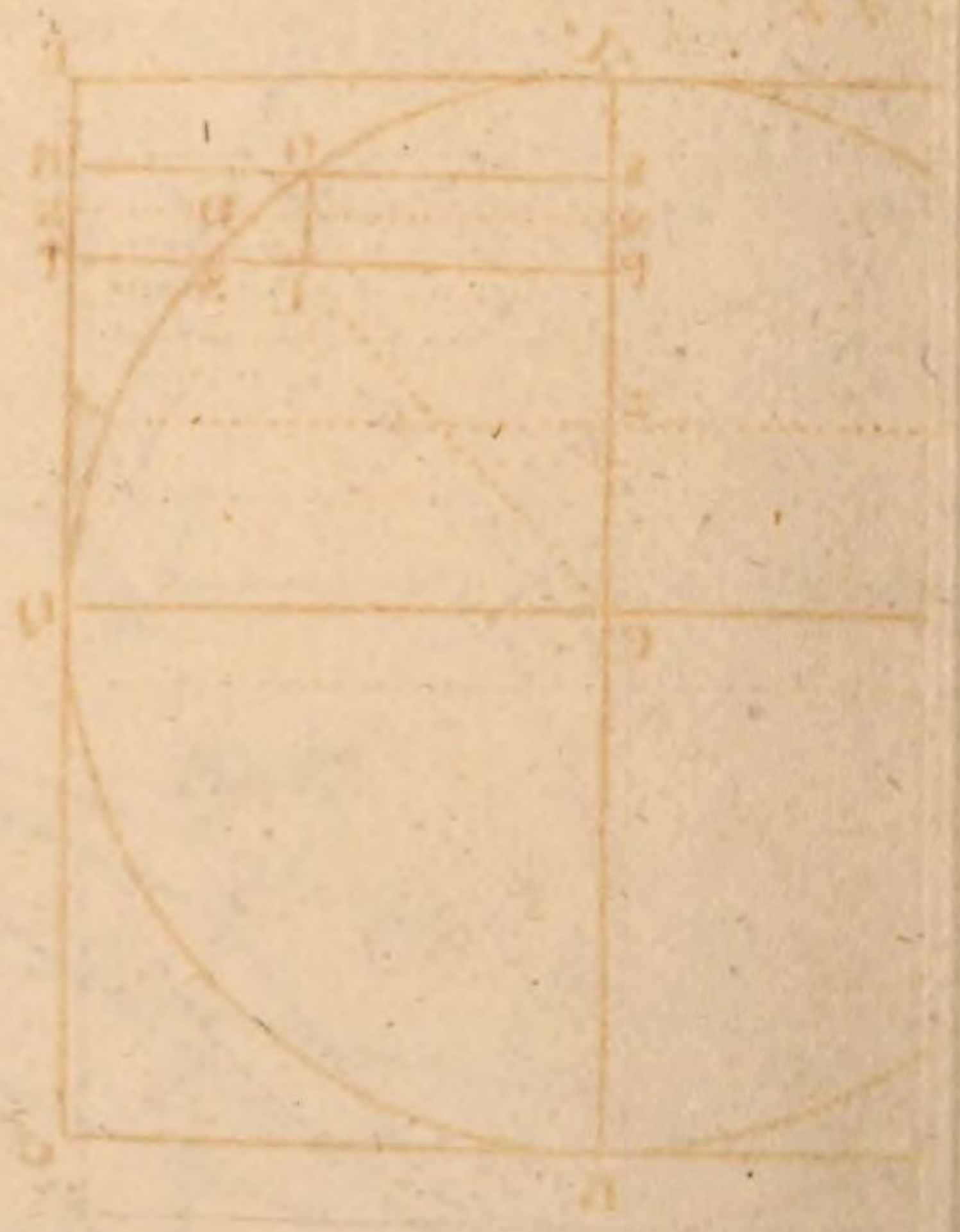
Fig. 63.











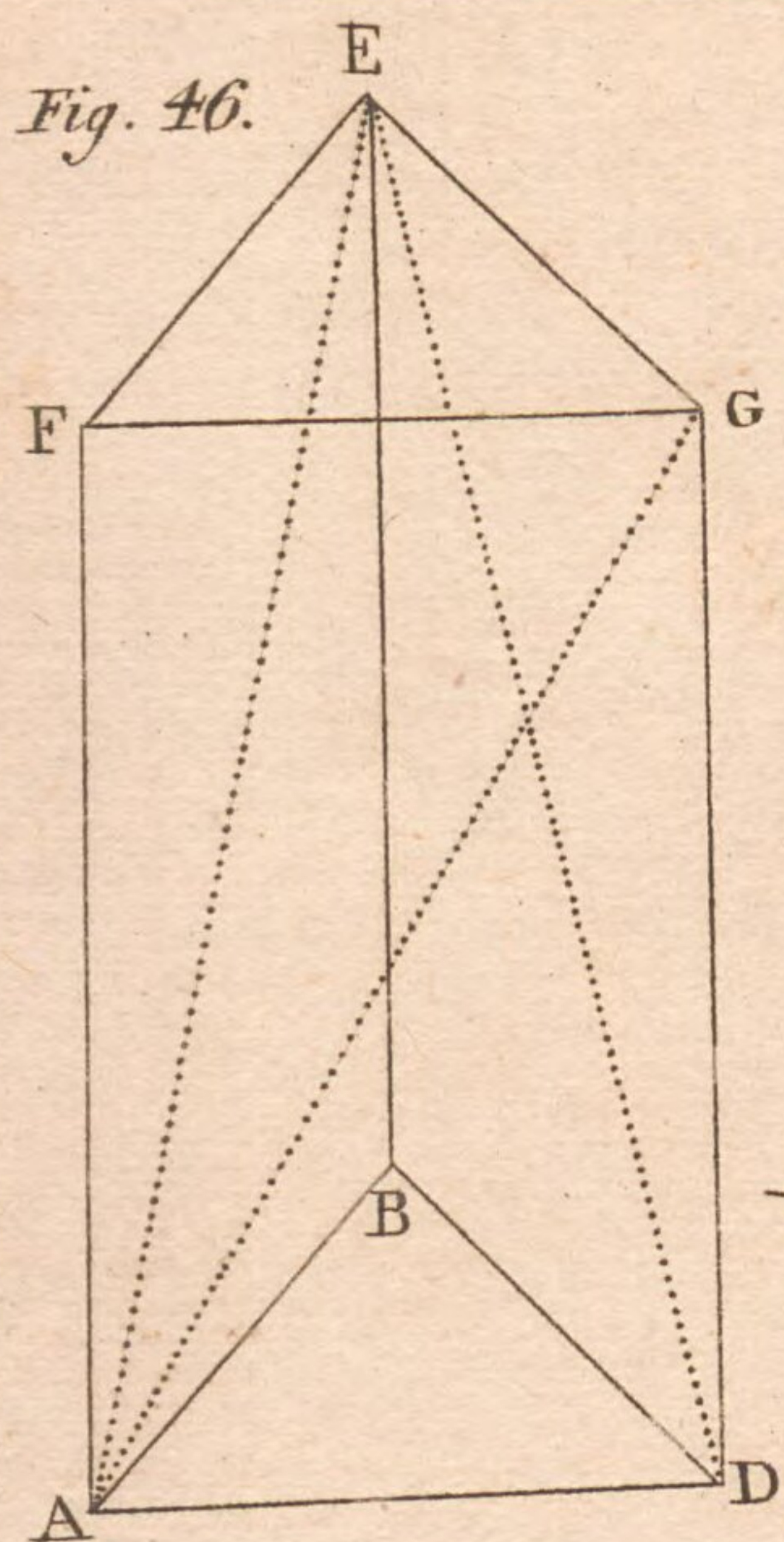


Fig. 46.

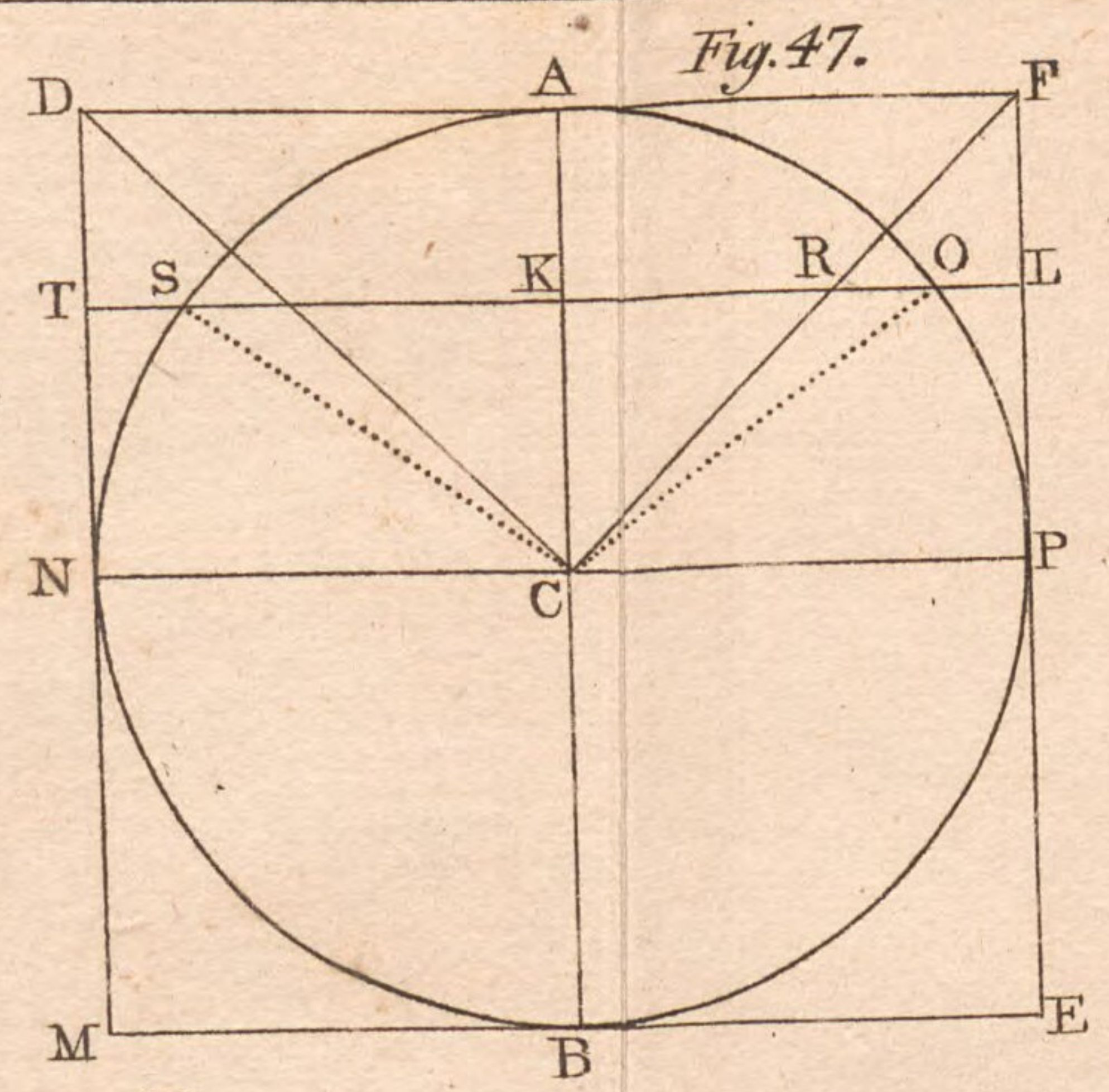


Fig. 47.

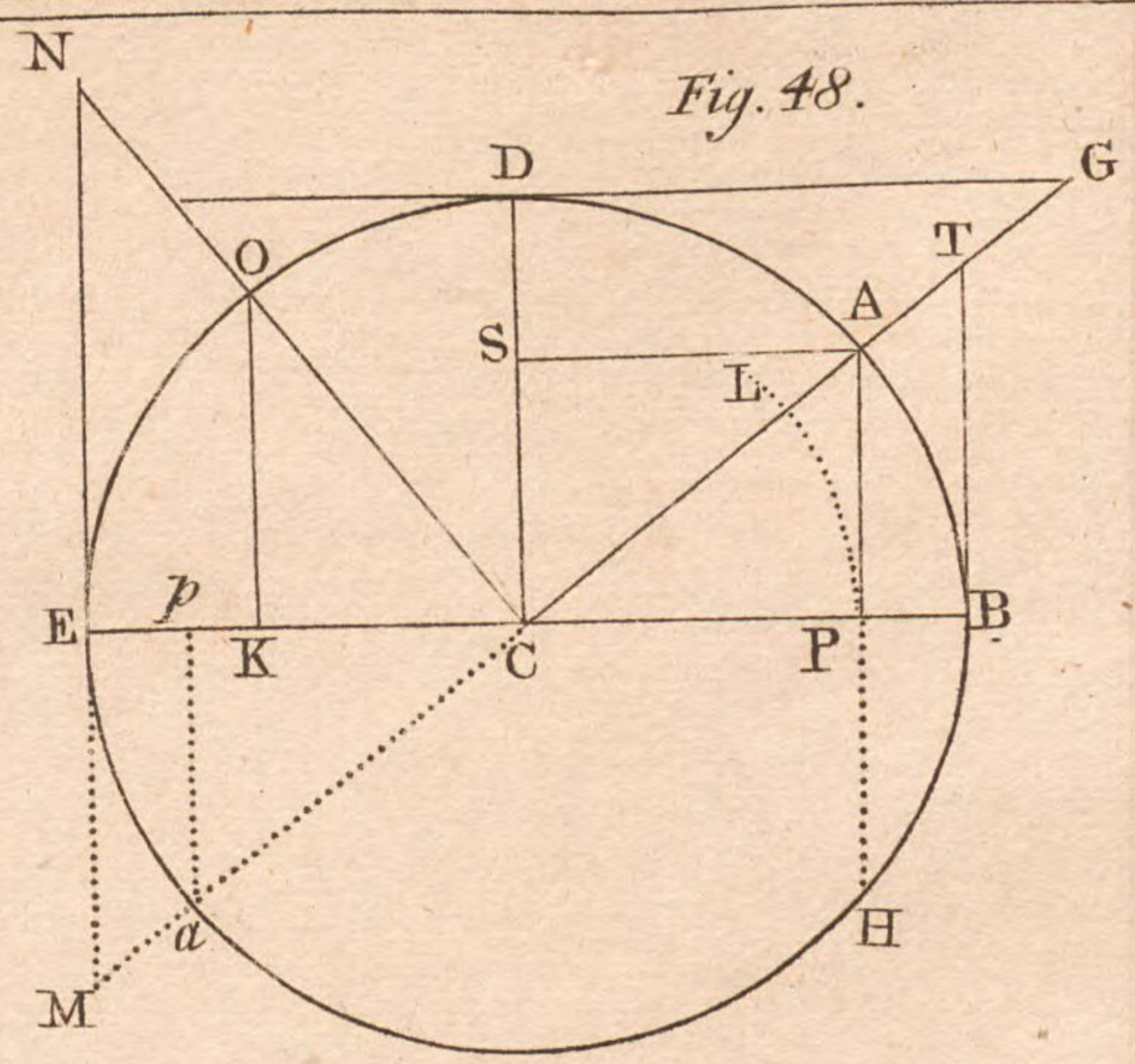


Fig. 48.

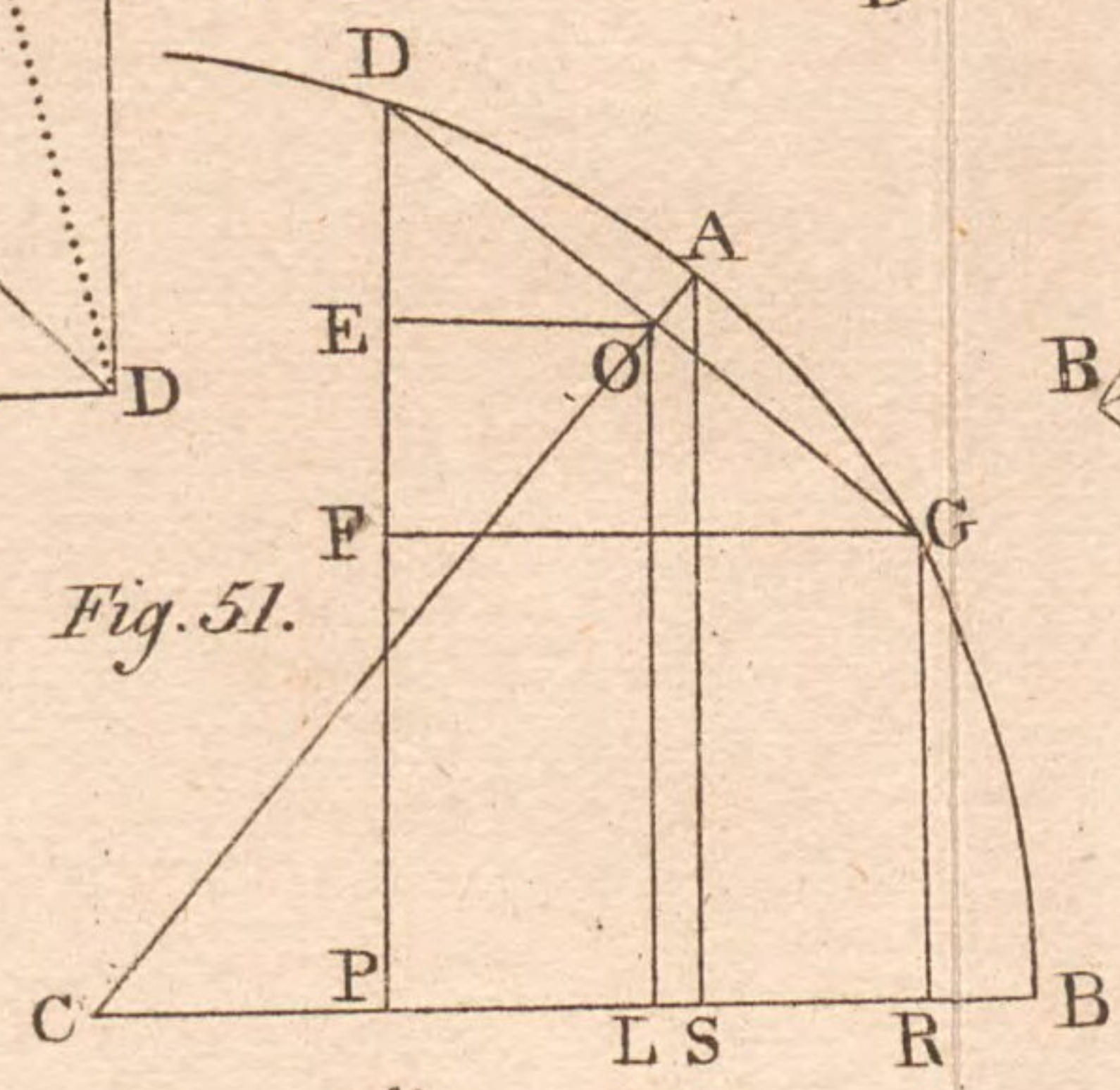


Fig. 51.

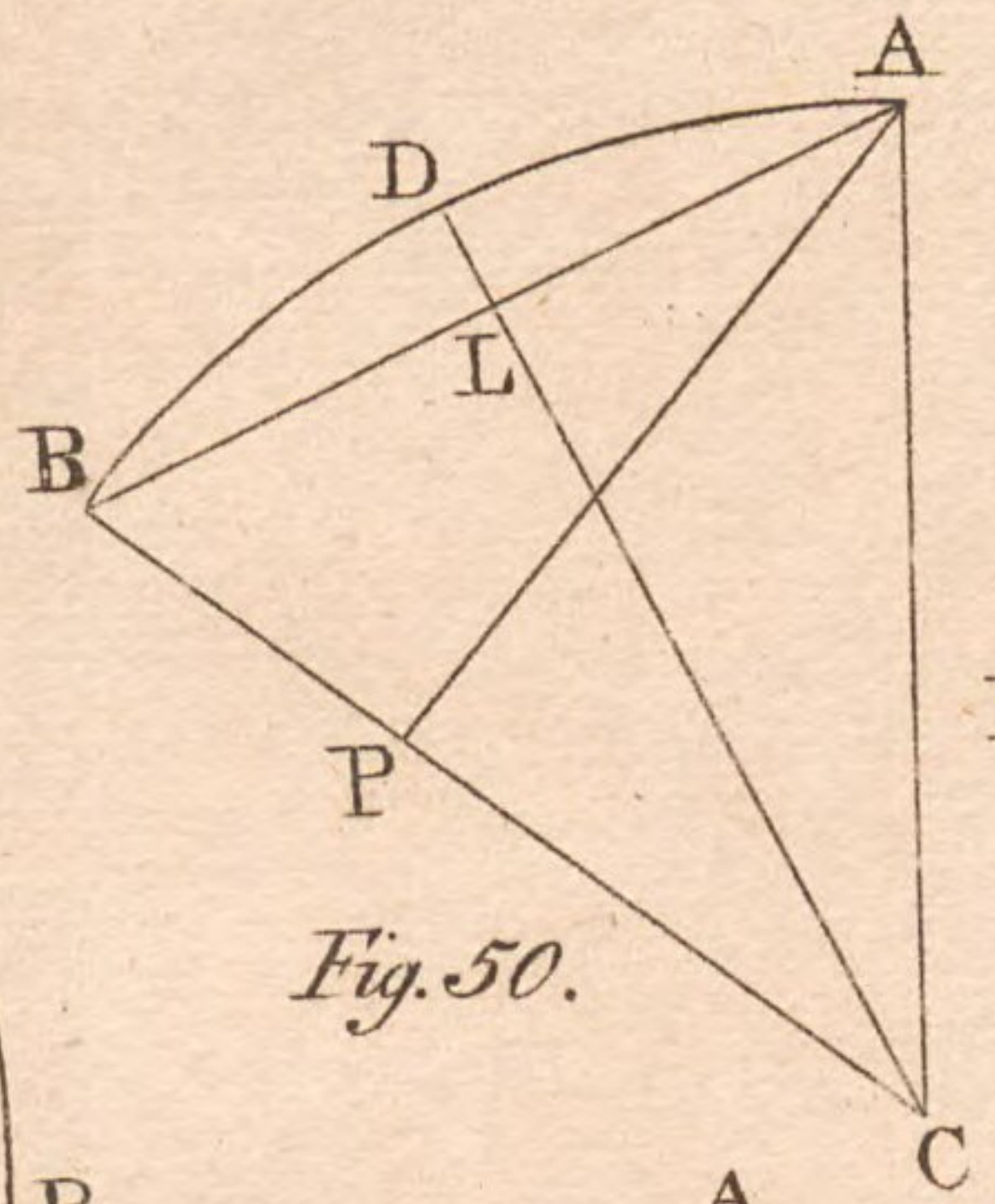


Fig. 50.

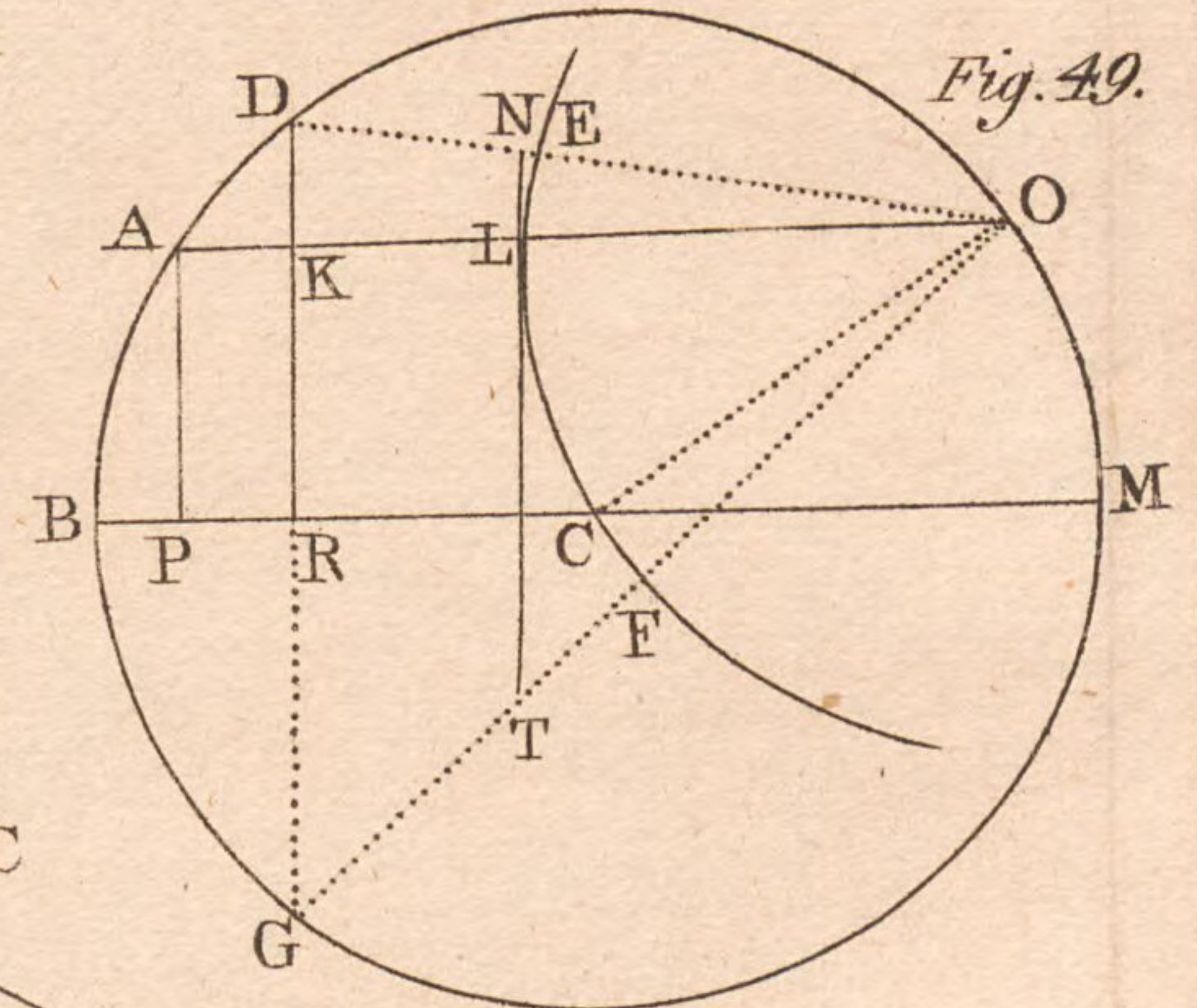


Fig. 49.

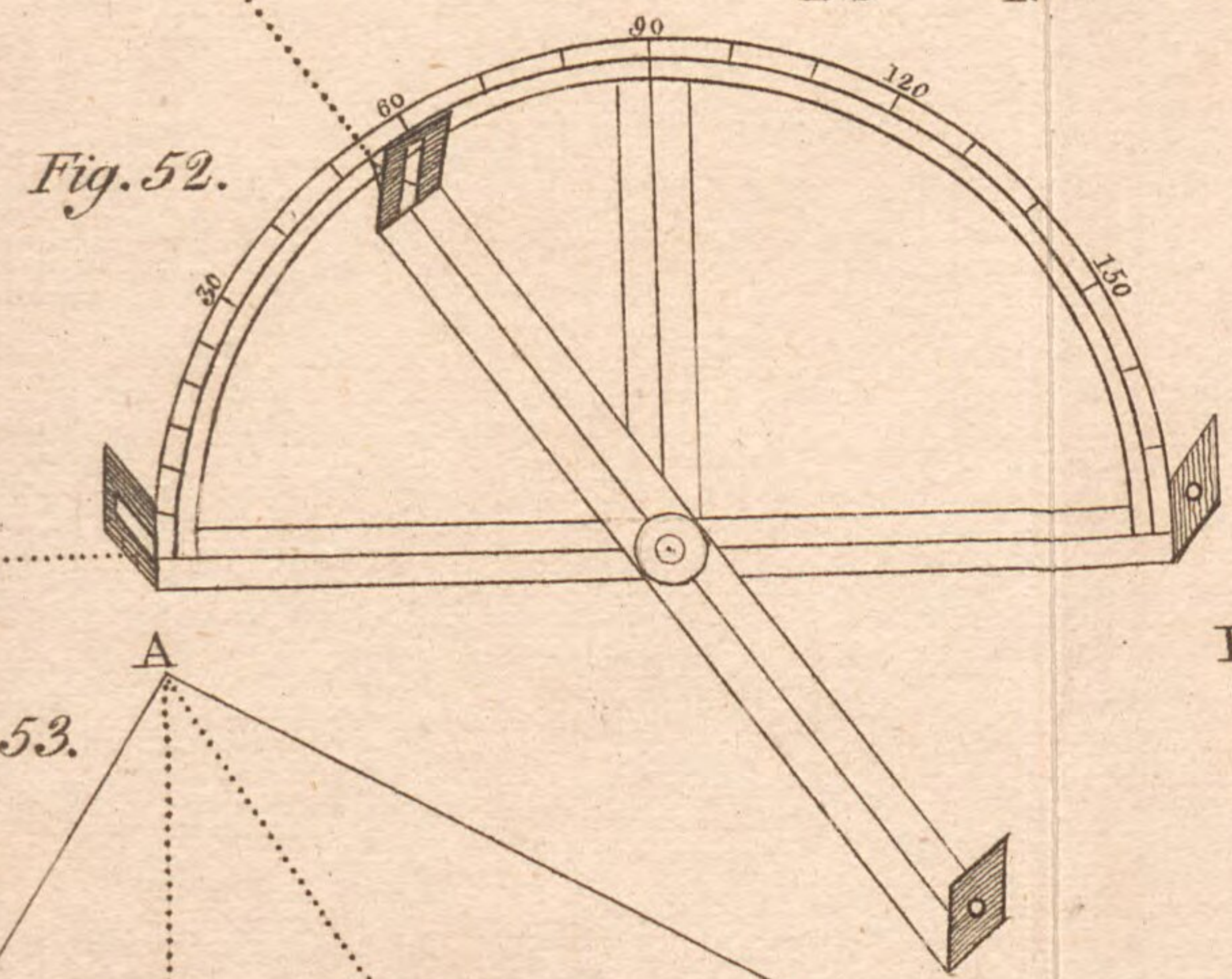


Fig. 52.

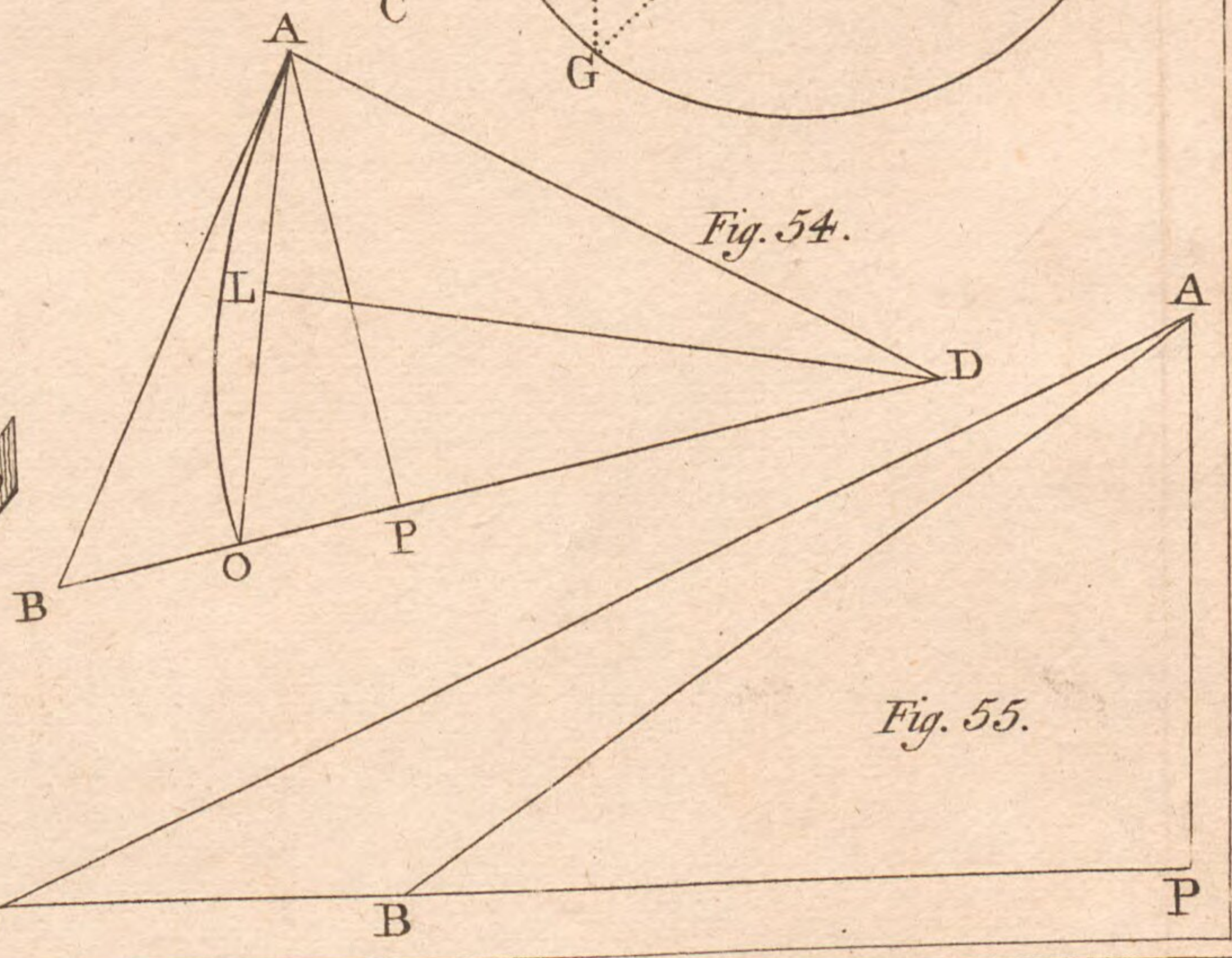


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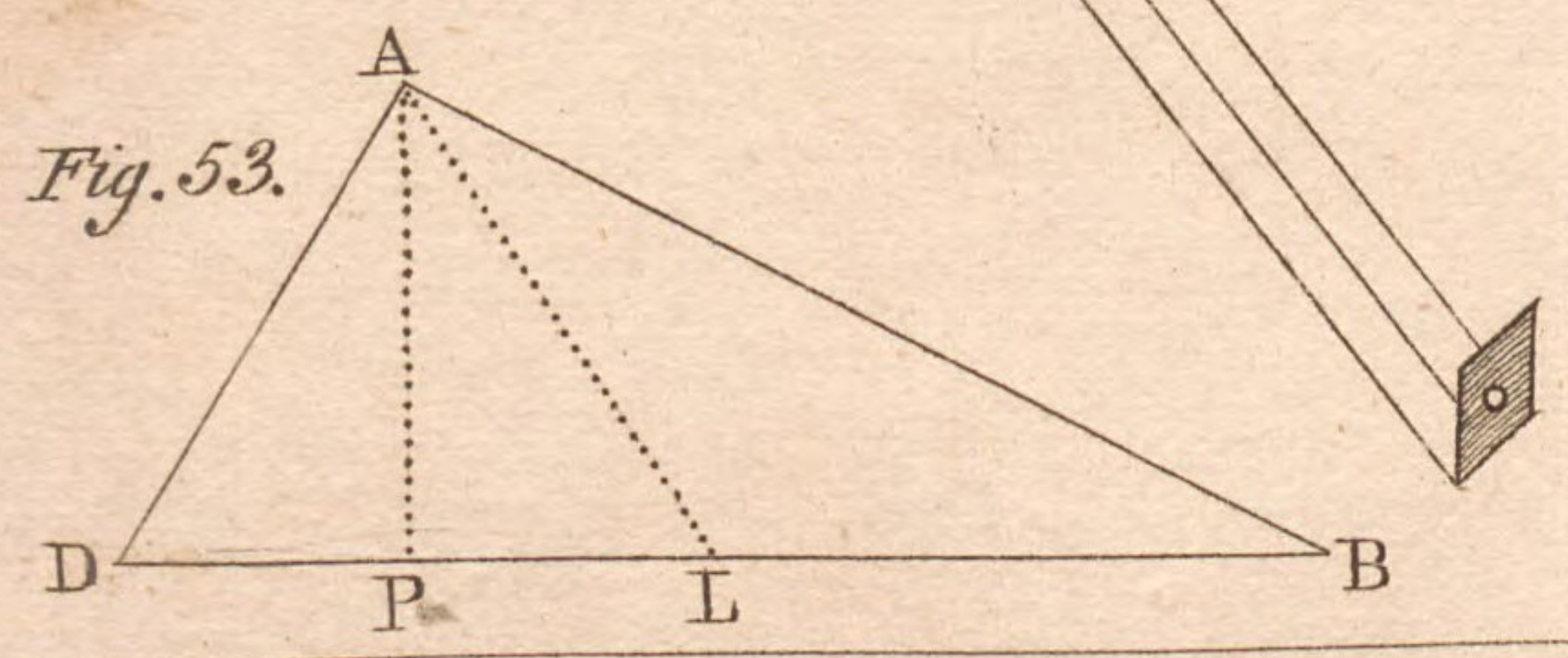


Fig. 53.

Fig. 55.



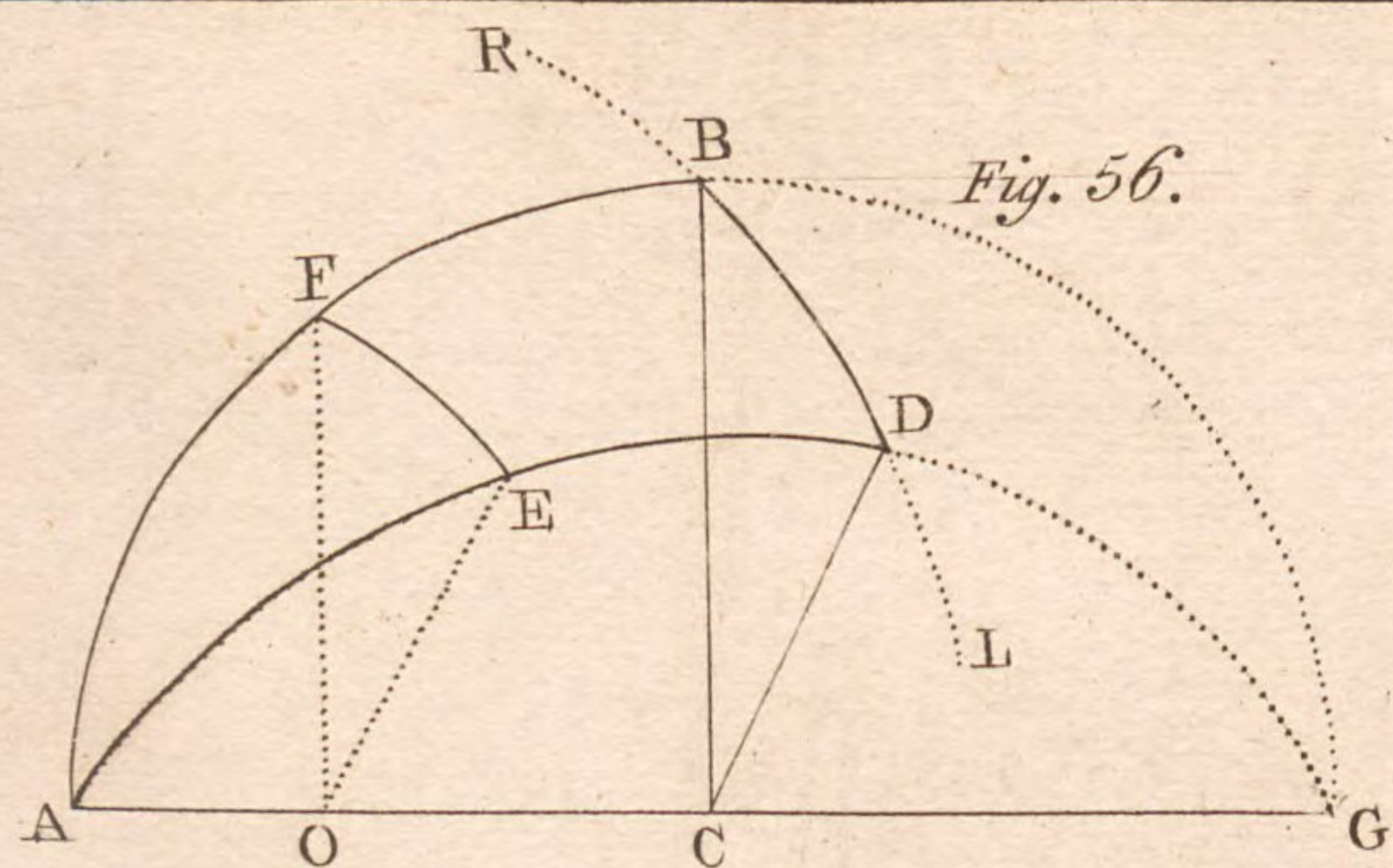


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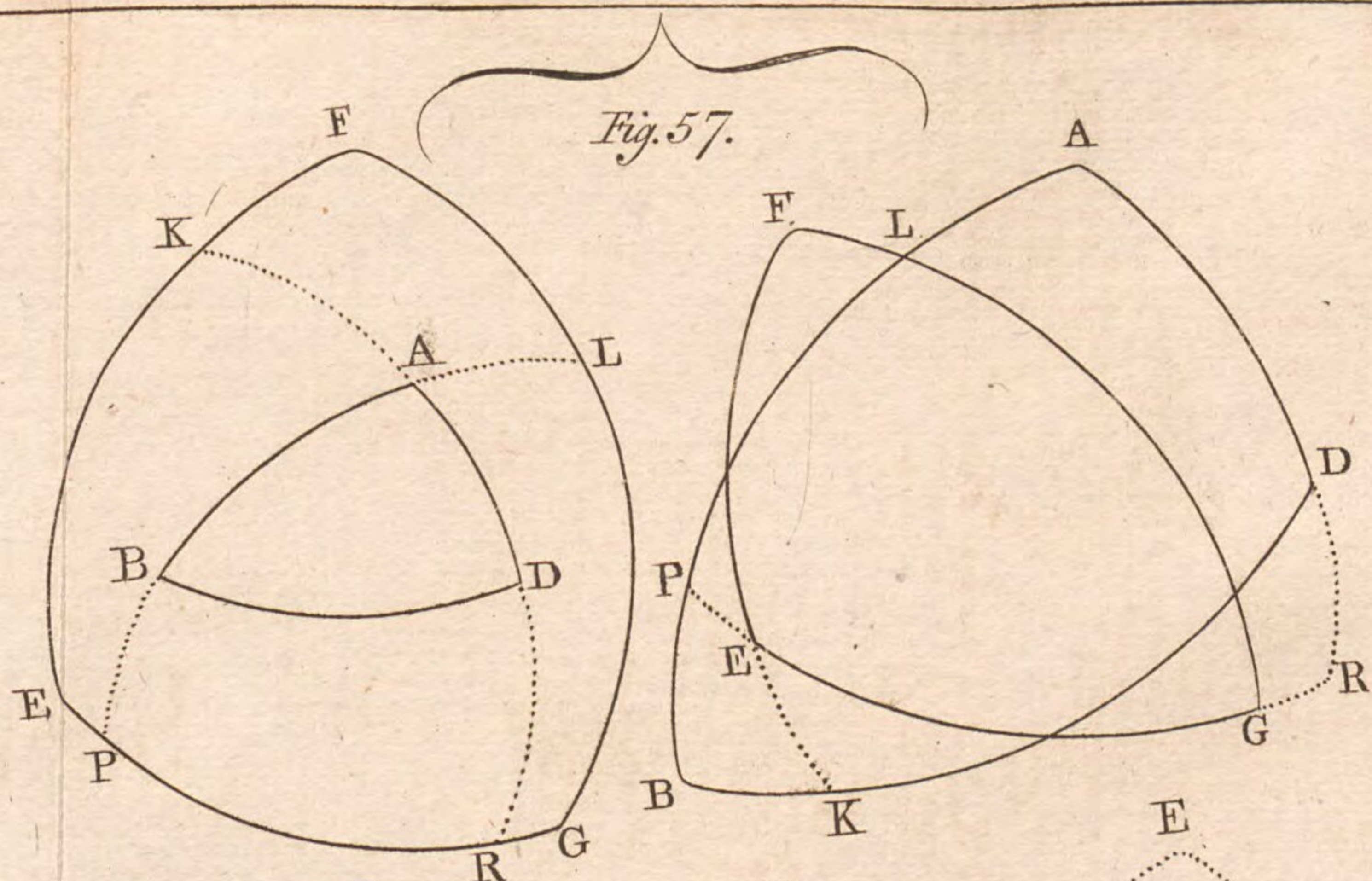


Fig. 57.

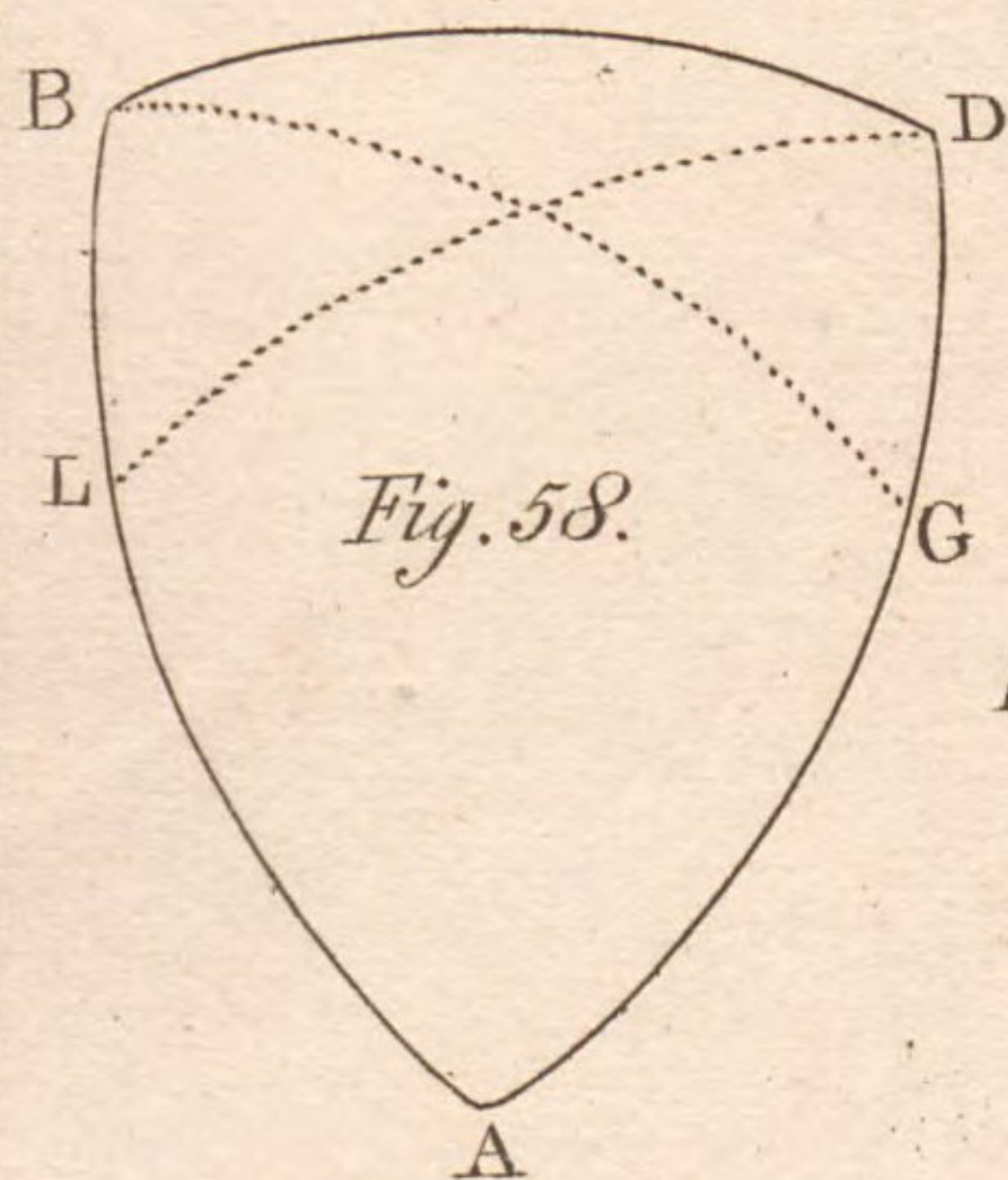


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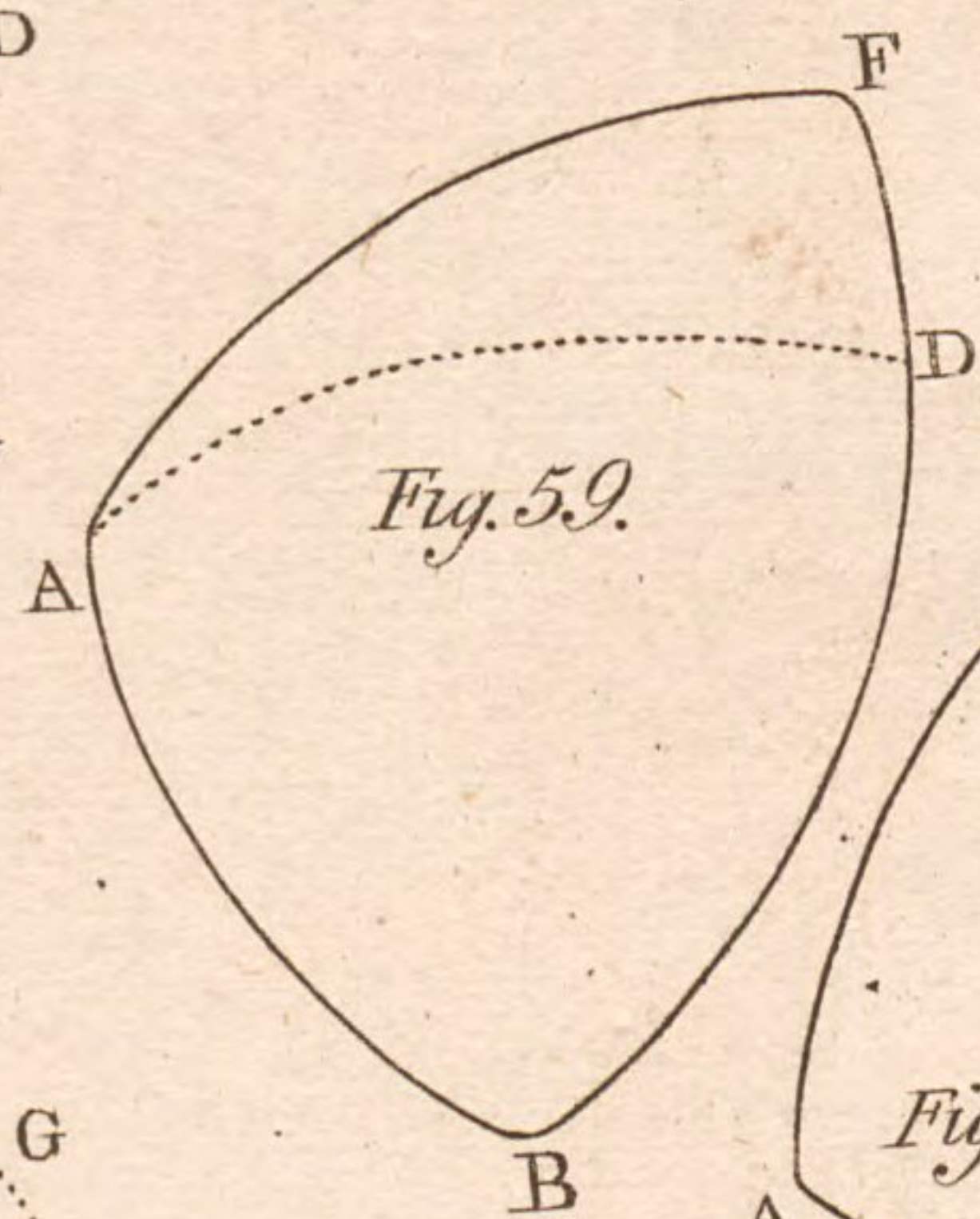


Fig. 59.

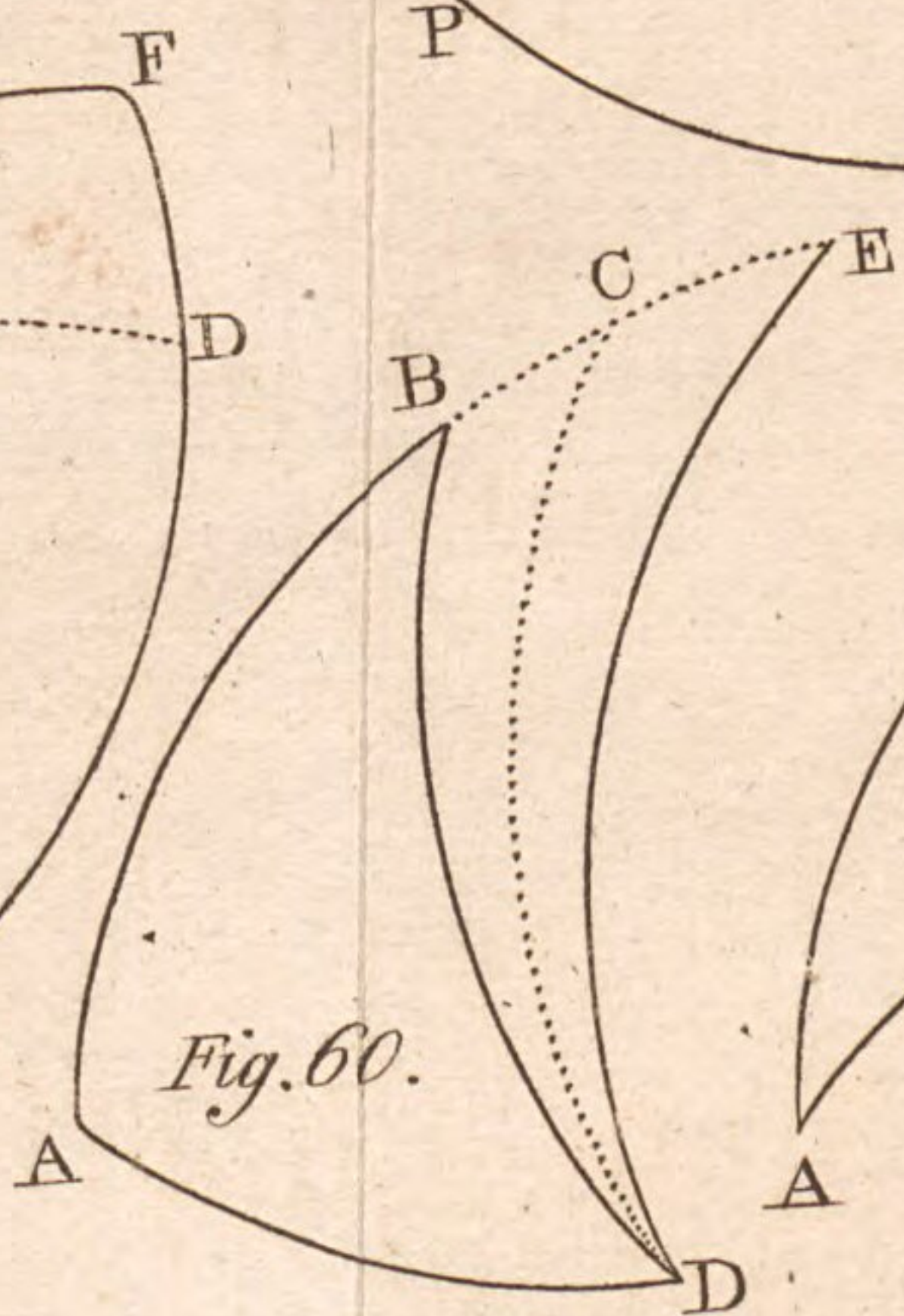


Fig. 60.

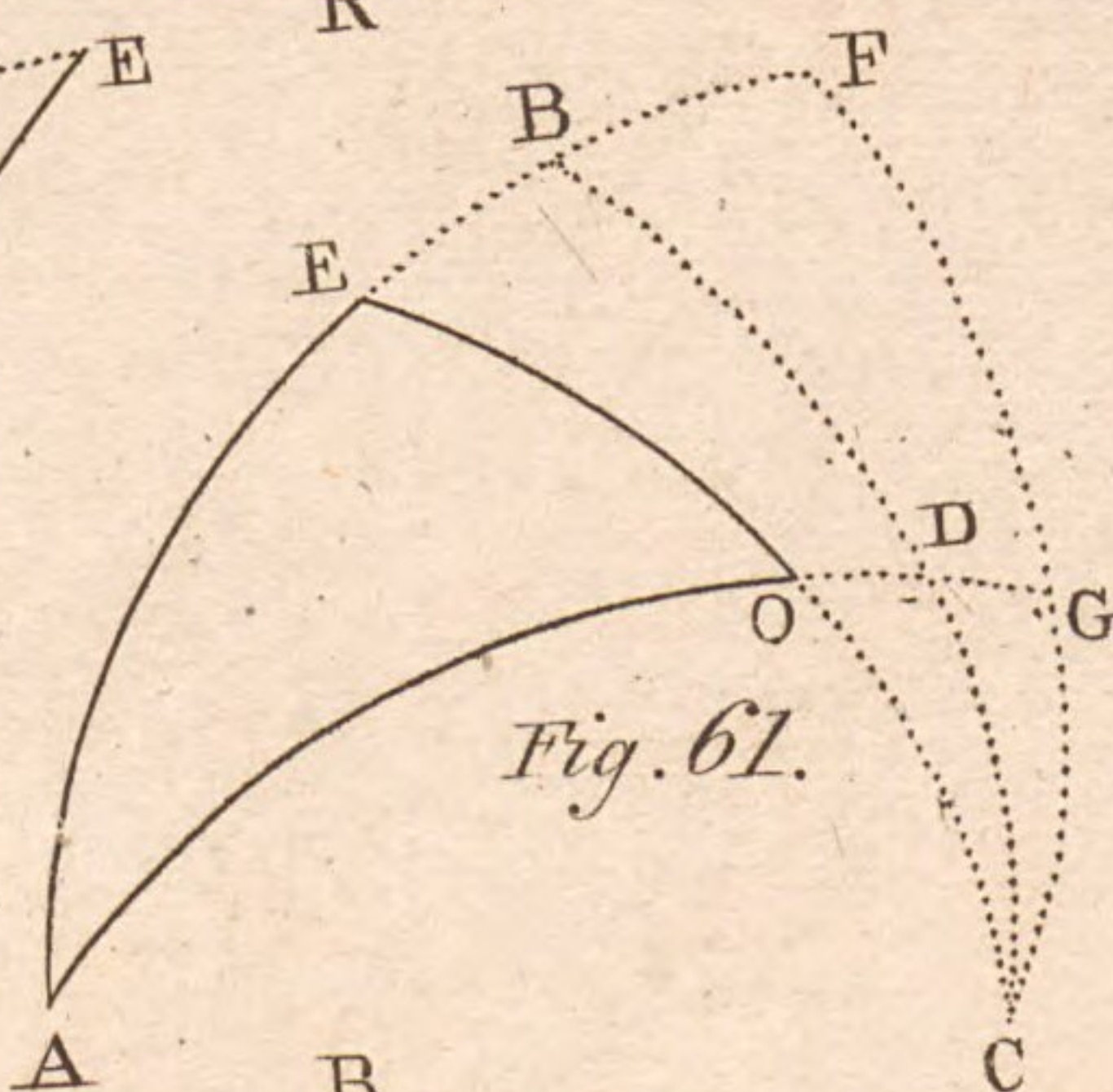


Fig. 61.

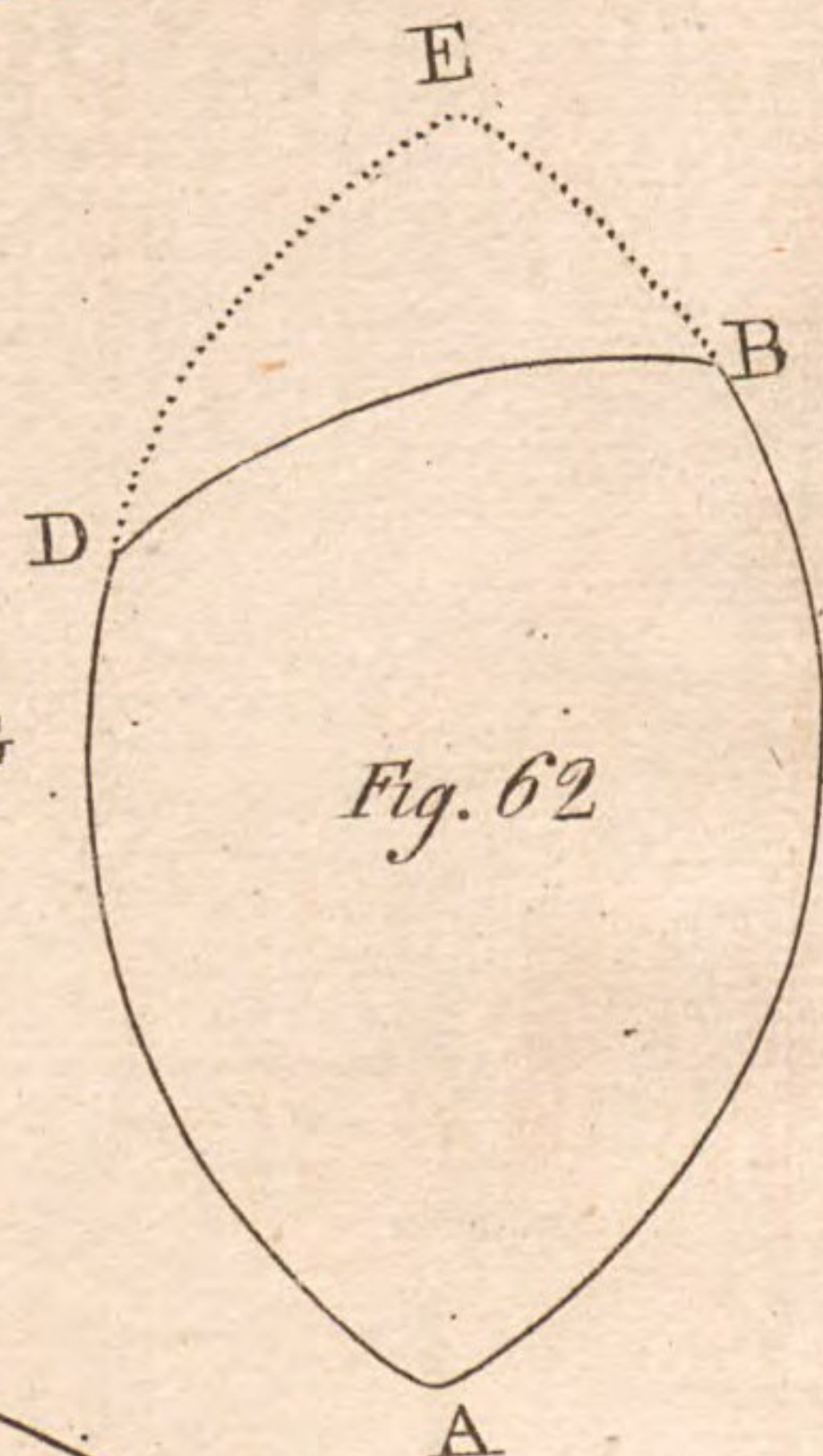


Fig. 62.

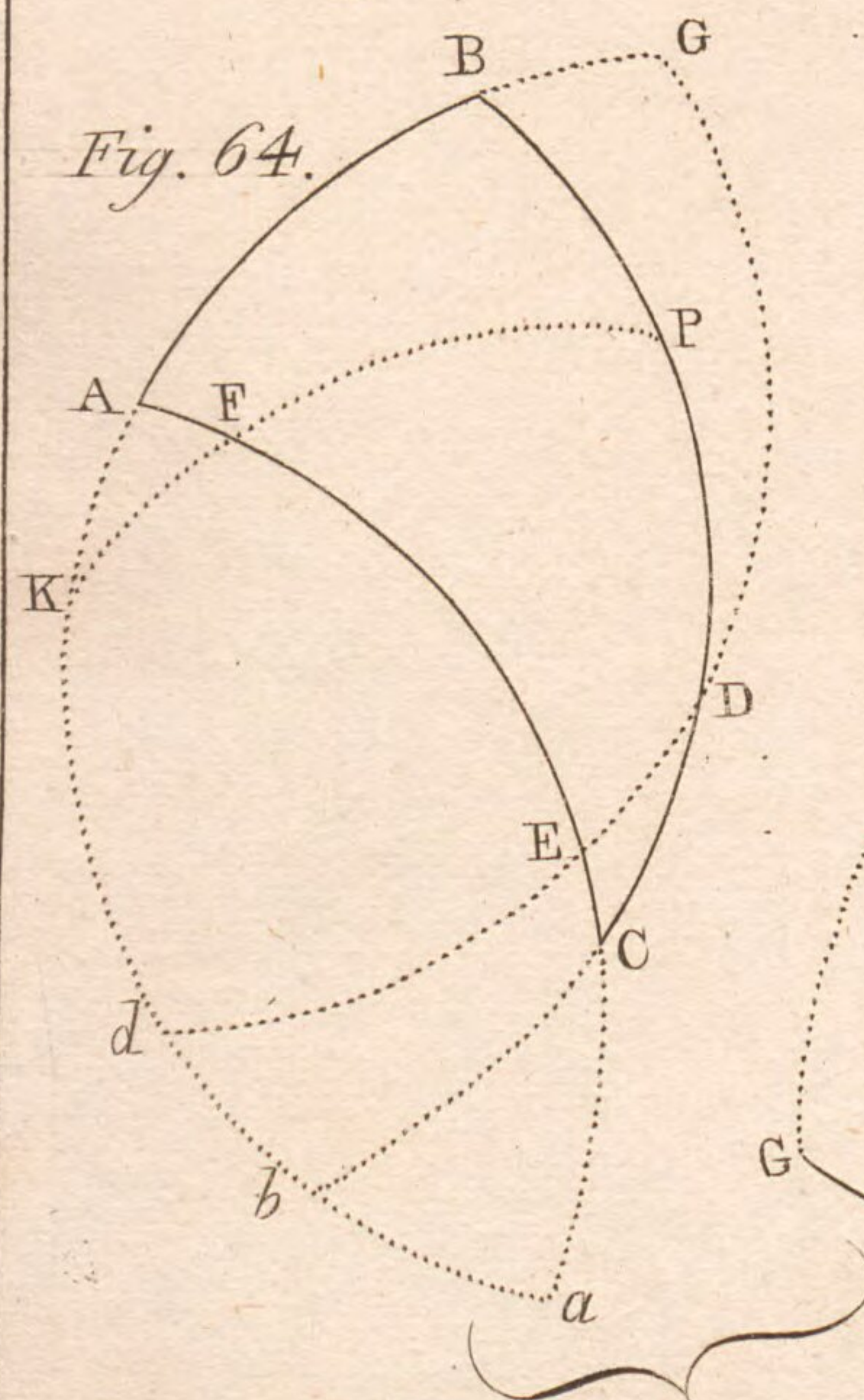


Fig. 64.

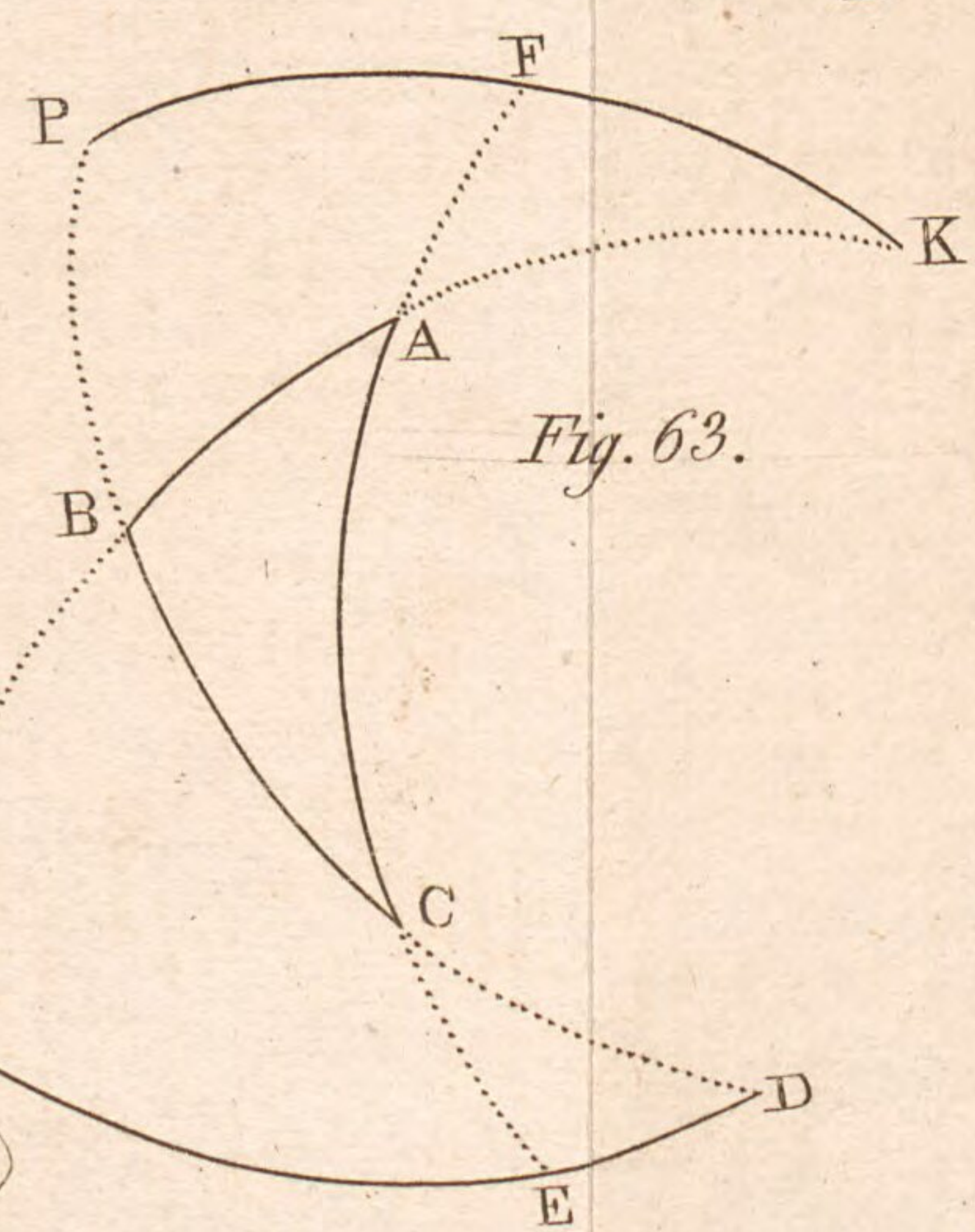


Fig. 63.

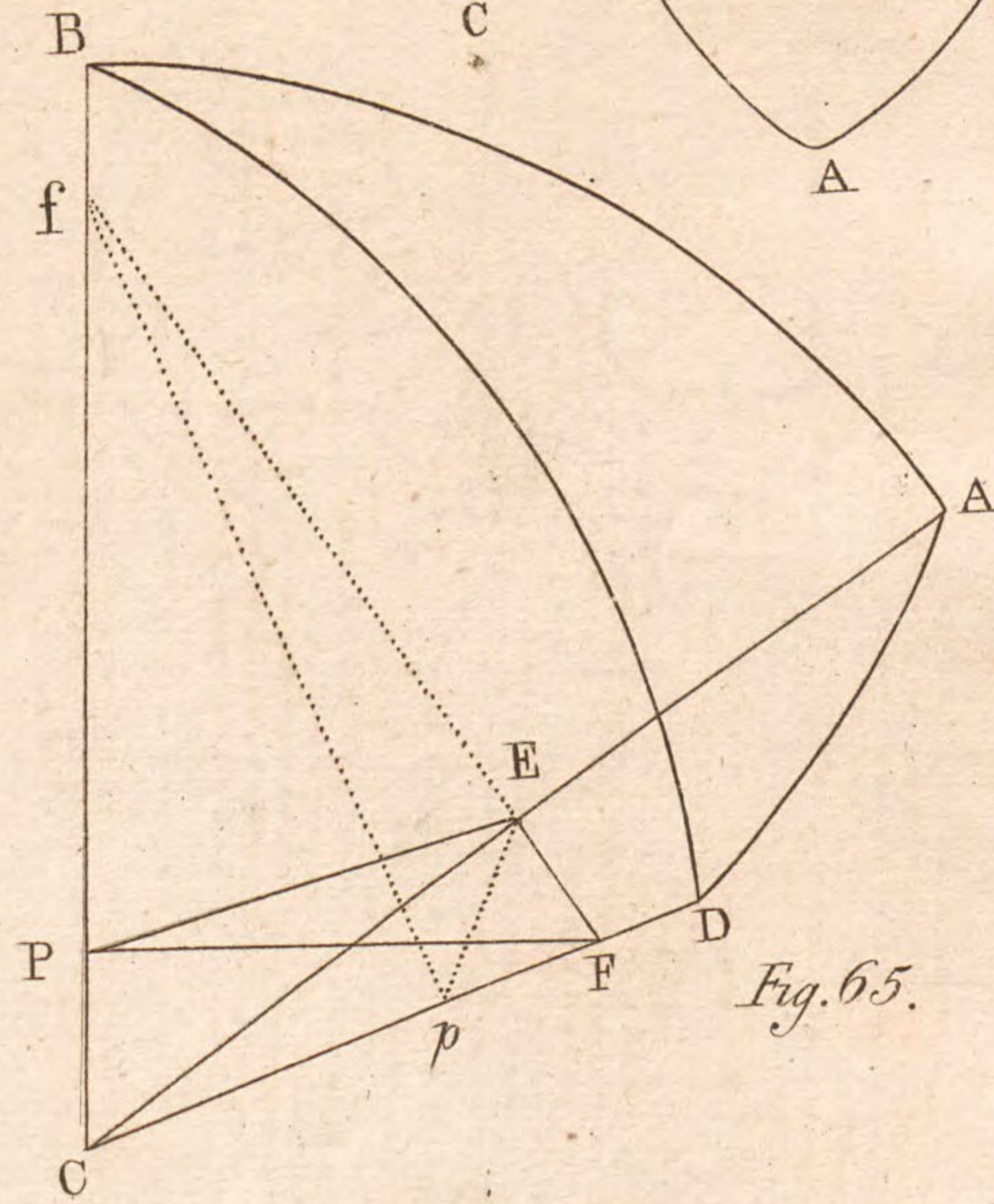
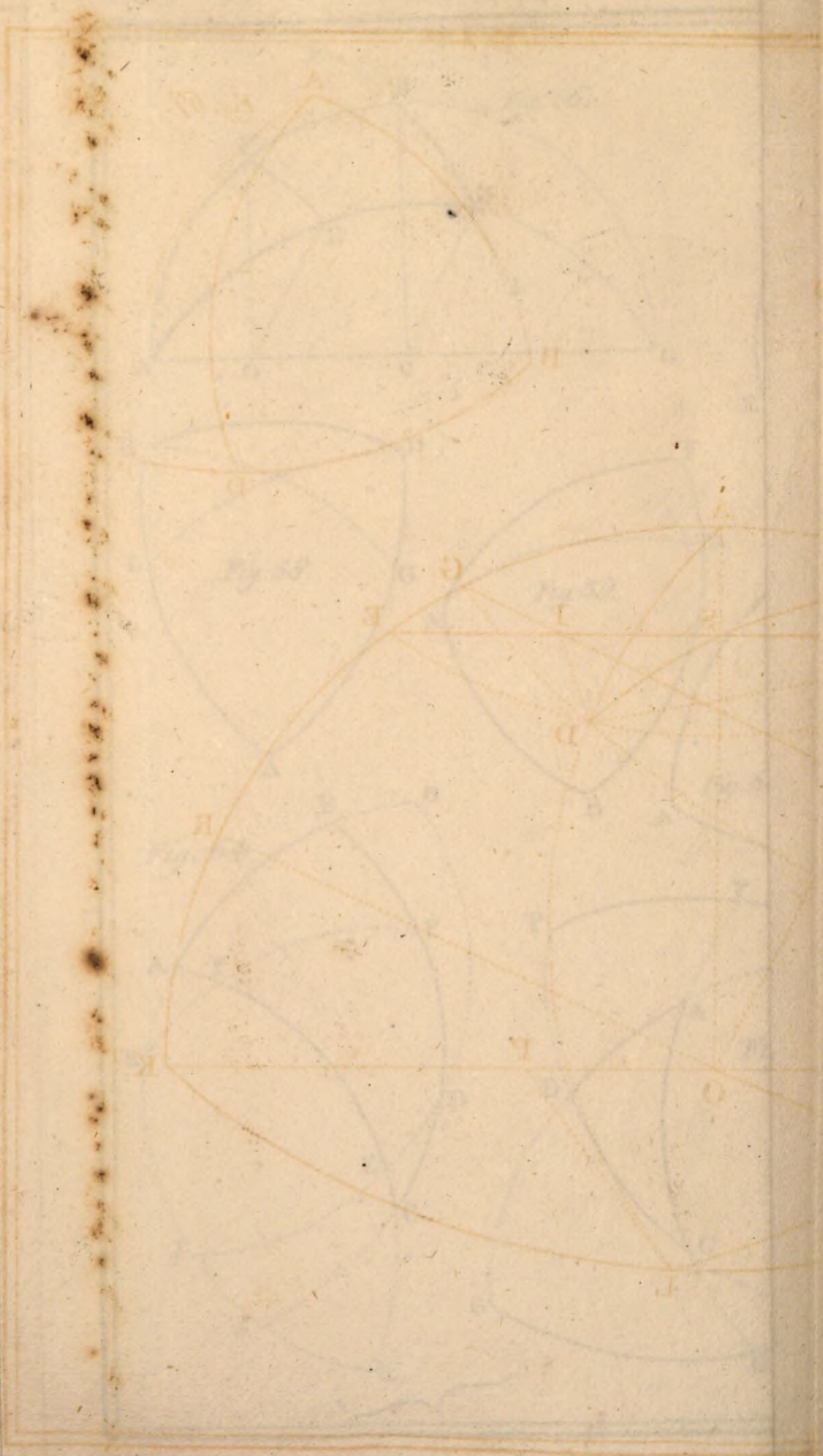
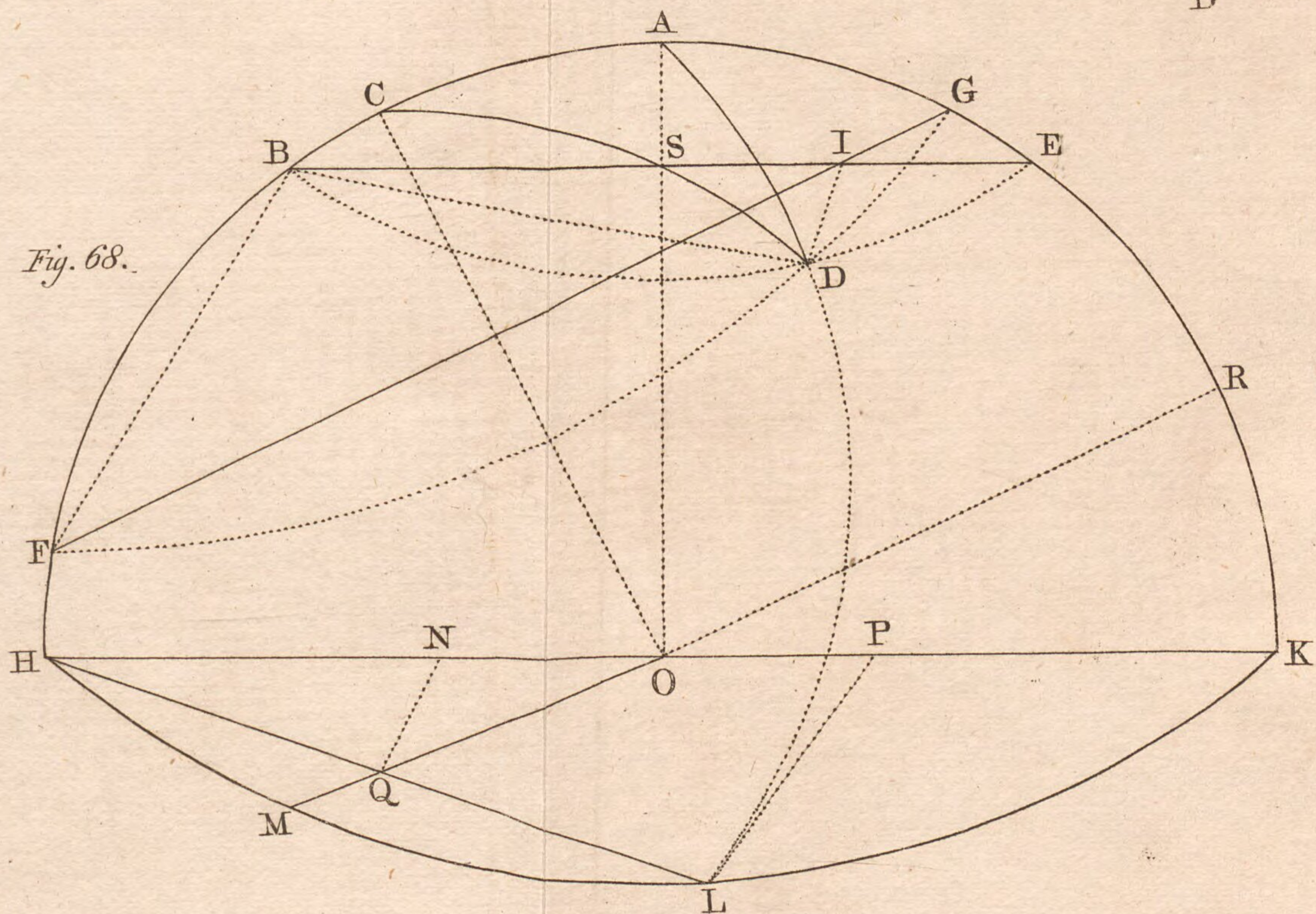
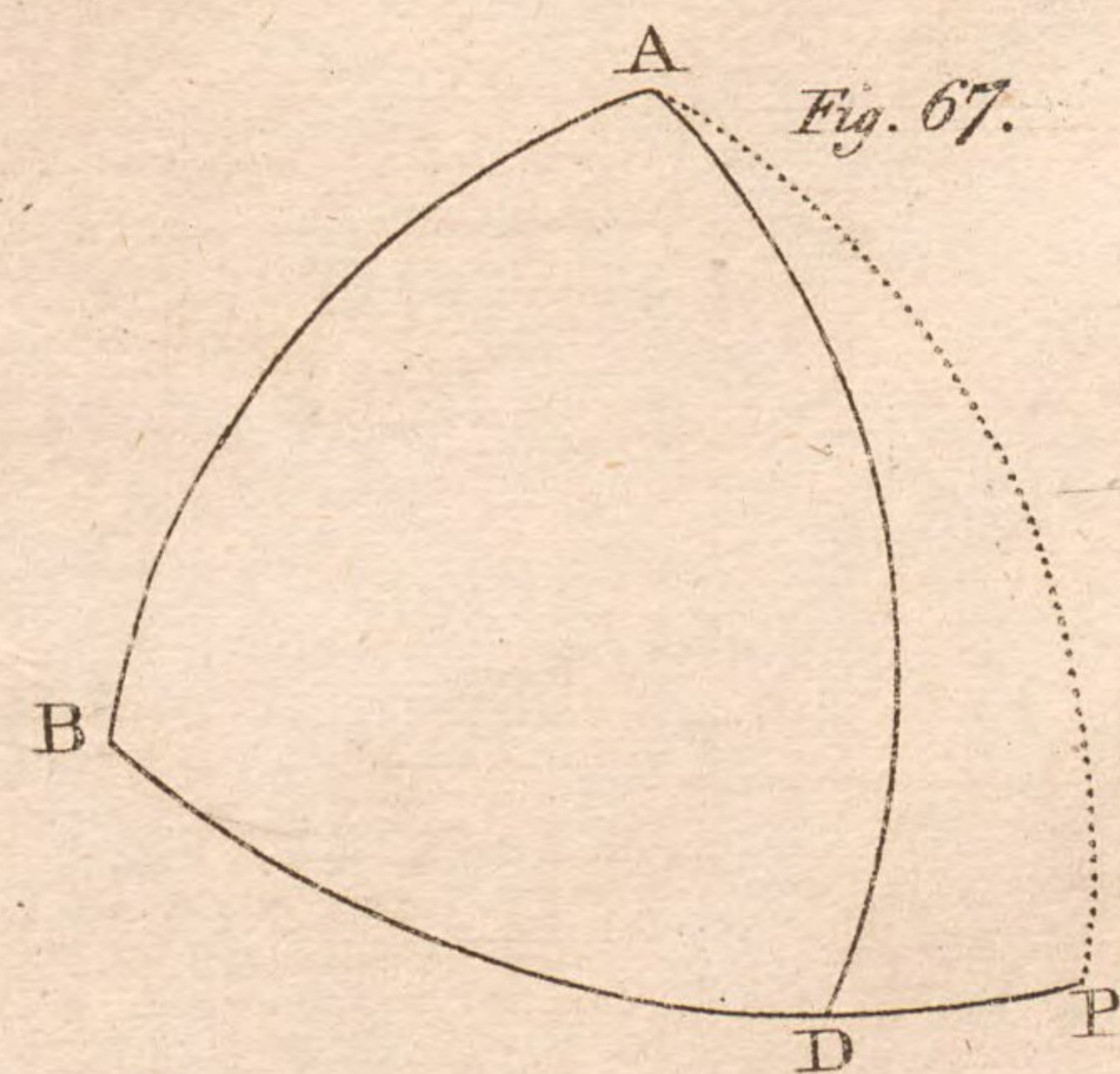
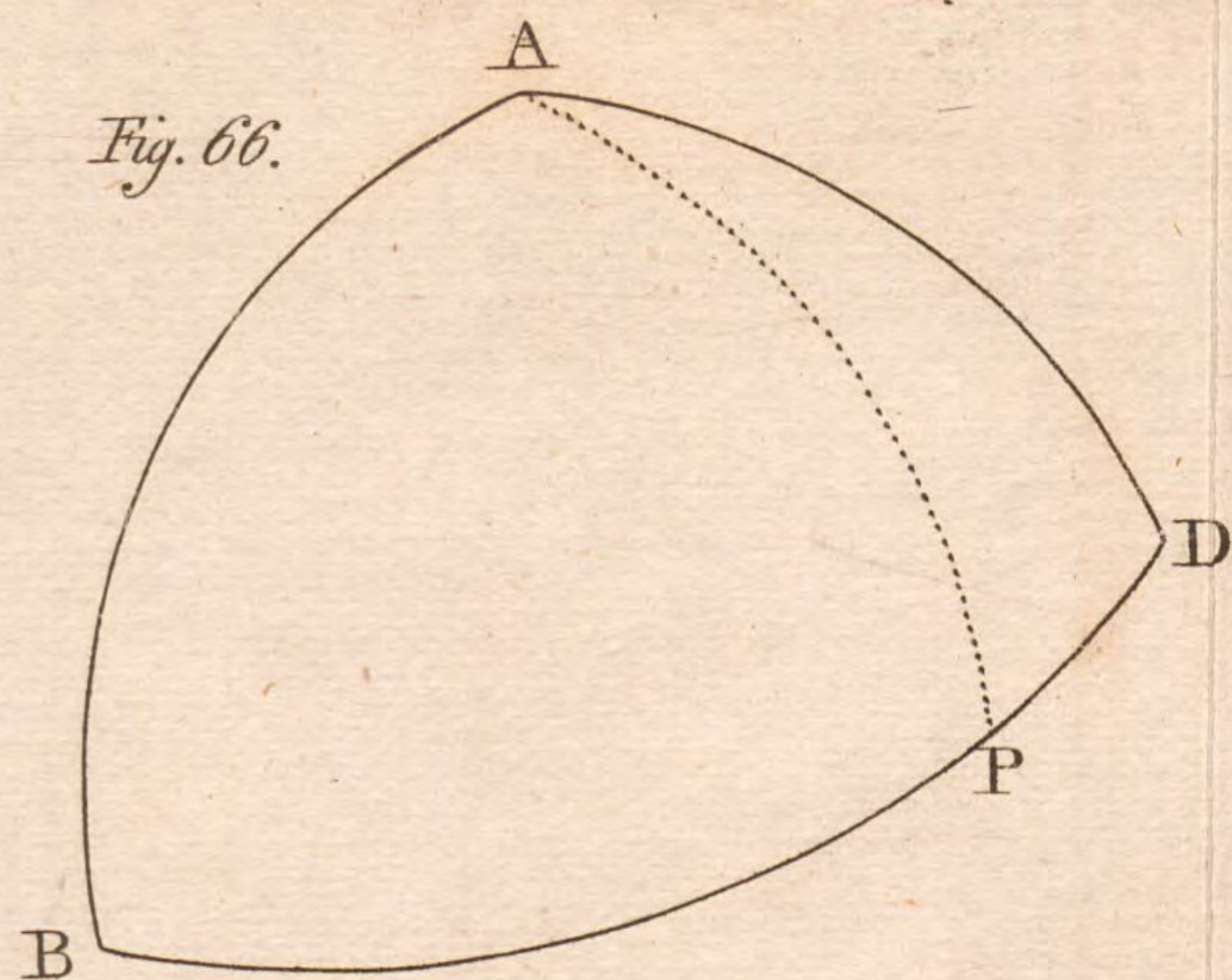
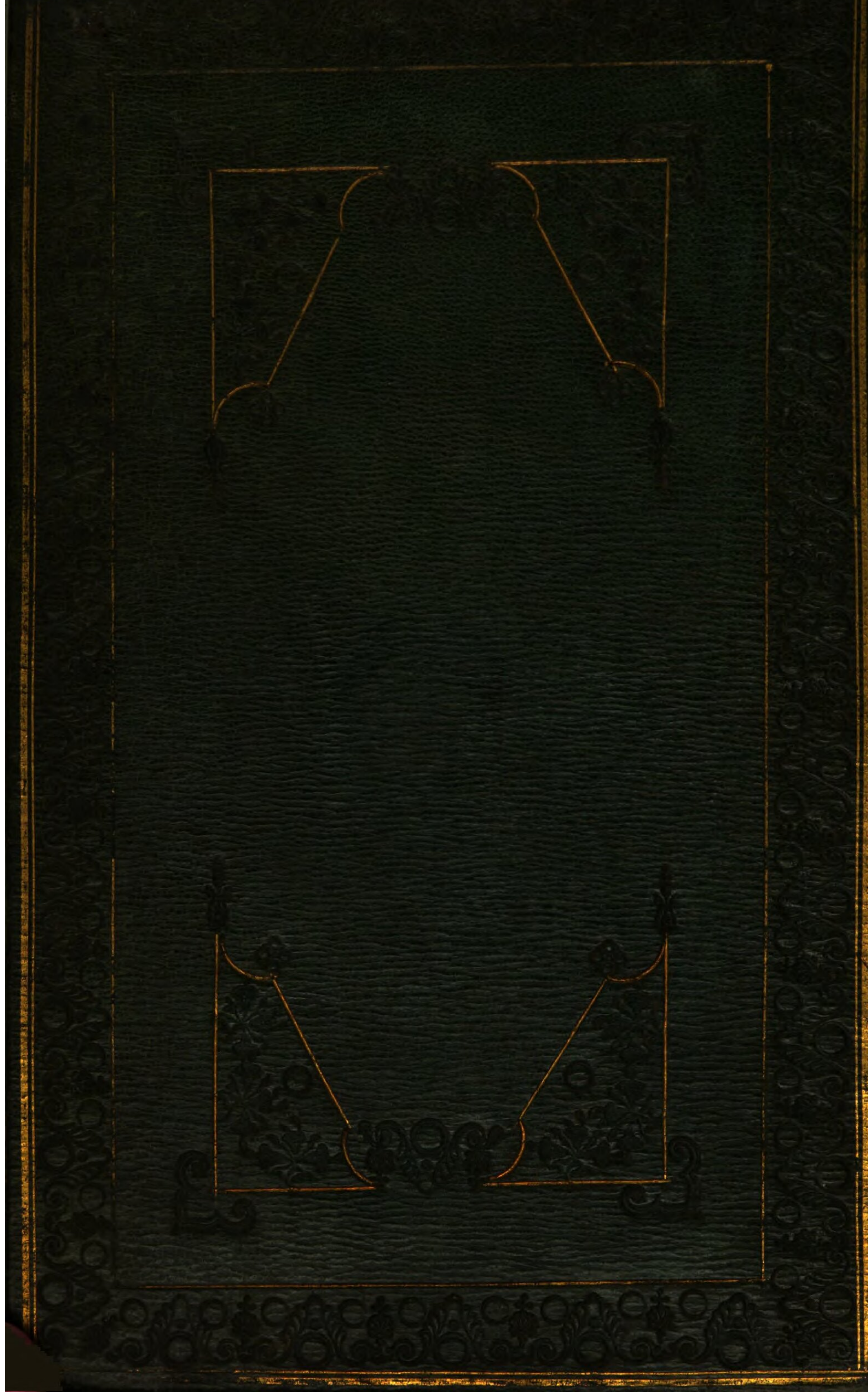


Fig. 65.







Darré, ...

Elements of geometry, with both plane and spherical trigonometry

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